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Reconstruction of sceneries with correlated colors

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Abstract

Matzinger (Random Structure Algorithm 15 (1999a) 196) showed how to reconstruct almost every three color scenery, that is a coloring of the integers $\mathbb{Z}$ with three colors, by observing it along the path of a simple random walk, if this scenery is the outcome of an i.i.d. process. This reconstruction needed among others the transience of the representation of the scenery as a random walk on the three-regular tree T_3. Den Hollander (private communication) asked which conditions are necessary to ensure this transience of the representation of the scenery as a random walk on T_3 and whether this already suffices to make the reconstruction techniques in Matzinger (1999a) work. In this note we answer the latter question in the affirmative. Also we exhibit a large class of examples where the above-mentioned transience holds true. Some counterexamples show that in some sense the given class of examples is the largest natural class with the property that the representation of the scenery as a random walk is transient.

Keywords: Scenery reconstruction; Random walks; Ergodic theory

1. Introduction

The following problems to which this paper will make a contribution were discovered in the context of ergodic theory, for example in connection with the so-called $T - T^{-1}$-problem (see Kalikow, 1982), and phrased as statistical questions independently by den Hollander and Keane (1986) and Benjamini and Weiss.

For our purposes we will consider the one-dimensional lattice $\mathbb{Z}$. Actually, the following problems make sense also for arbitrary graphs, but as there are hardly any results apart from the case when this graph is $\mathbb{Z}^d$ for some $d \in \mathbb{N}$, we immediately

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concentrate to our object of desire. Assume that $\mathbb{Z}$ is colored with m colors. More precisely, we consider two such colorings, that is we consider two functions

$$\eta, \xi : \mathbb{Z} \to \{0, \dots, m-1\}$$

and call these functions m-color sceneries or simply sceneries. Let $(S_k)_{k \in \mathbb{N}_0}$ be symmetric and simple random walk on $\mathbb{Z}$ starting in the origin and walking without holding, that is $S_0 = 0$ and

$$P(S_{k+1} = x + 1 | S_k = x) = P(S_{k+1} = x - 1 | S_k = x) = \tfrac{1}{2}$$

for all $x \in \mathbb{Z}$ and $k \in \mathbb{N}_0$. Moreover, define $\chi := (\chi_k)_{k \in \mathbb{N}_0}$ to be the color record of $(S_k)_{k \in \mathbb{N}_0}$, that is either

$$\chi_k = \xi(S_k) \quad \text{for all } k \text{ or} \quad \chi_k = \eta(S_k) \quad \text{for all } k$$

depending on which scenery we observe the colors. The question now is: can we just by observing χ (and, of course, without any further knowledge of $(S_k)_{k \in \mathbb{N}_0}$) tell on which of the sceneries ξ or η this color record χ has been produced?

Remarkable answers to this question (even for the higher dimensional case) have been given by Benjamini and Kesten (1996), who showed that if ξ and η are produced by an i.i.d. process on $\mathbb{Z}$ (that is to say, if $\xi(z)$ and $\eta(z), z \in \mathbb{Z}$ are i.i.d. random variables), then $\xi(z)$ and $\eta(z)$ can almost surely be distinguished by their color record, if the dimension $d = 1, 2$ and $m \geqslant 2$ is arbitrary. More precisely in this situation, there exists a test which tells with probability one on which of ξ and η the color record χ has been produced (even with a slightly stronger version of distinguishability excluding trivial solutions such as benefiting from the fact that e.g. $\xi(0) \neq \eta(0)$).

Also Kesten (see Kesten, 1996) showed that in dimension one, if $m \geqslant 5$, and ξ is again i.i.d. we can almost surely detect a single defect in ξ from knowing χ, that is, we can almost surely tell, whether χ has been produced on ξ or a scenery η differing from ξ in one vertex $i \in \mathbb{Z}$, only. Here and in the following the notion "almost surely" will refer to a probability measure $\mathbb{P}$ describing both, the randomness in $(S_k)_{k \in \mathbb{N}_0}$, and the randomness in ξ (or in ξ and η, if we are interested in two sceneries) and making $(S_k)_{k \in \mathbb{N}_0}$ and ξ (or $(S_k)_{k \in \mathbb{N}_0}$, η, and ξ, respectively) independent.

Indeed even more is true: In dimension one Matzinger (1999a, b) showed that for arbitrary $m \geqslant 2$ one can even almost surely reconstruct ξ from $\chi = (\xi(S_k))_{k \in \mathbb{N}_0}$, that is one can reproduce a scenery ξ' from χ which is equal to ξ up to translation and reflection at the origin. This is even true, if the underlying random walk S is allowed to jump (if the jumps are bounded and m is large enough). The latter was shown recently by the authors in collaboration with Merkl (Löwe et al., submitted).

All these results are particularly surprising, since on the other hand it is known that there are uncountably many sceneries which cannot be distinguished by their color record. This has been proven by Lindenstrauss (1999).

The analogue to the reconstructability of ξ from χ in two dimensions has been recently proven by the authors under the conditioned that m is large enough (see Löwe and Matzinger, 2002).

Basically all reconstruction and distinction techniques cited above (with one exception) seem to strongly exploit the fact that the scenery is an i.i.d. process, that is that $\xi(z)$, $z \in \mathbb{Z}$ are i.i.d. random variables. Only the methods employed by Matzinger (1999a) are partially combinatorial and therefore seem to allow for a generalization to other sceneries. Indeed, den Hollander (private communication) asked what conditions for the scenery would be necessary to make the reconstruction ideas in Matzinger (1999a) work. We consider an answer to this question interesting in its own rights as it sheds some light on the universality of the solution to the above problem. In particular, the roots of the scenery reconstruction problem in ergodic theory make it attractive to ask for the ergodic properties of the sceneries needed to ensure reconstructability. Moreover, it was pointed out to us that similar ideas might be useful in the context of DNA reconstruction. As a DNA sequence usually is assumed to be Markov-dependent of some type (at least the assumption of i.i.d. letters is rather far fetched) an analysis of what kind of sceneries are reconstructible might also be helpful concerning possible applications.

This paper is divided into five sections. Section 2 contains a description of the basic setup and the first central result of this paper. In Section 3 we describe the fundamental reconstruction algorithm, while Section 4 contains a proof that the algorithm actually works under the conditions of Theorem 2.1. Finally in Section 5 we give the most important examples where Theorem 2.1 applies and also cases where it does not apply and actually reconstruction along the ideas of this paper is not possible. Furthermore, we indicate that for these examples not only we can reconstruct a randomly chosen scenery ξ, but also that there is a very powerful test for the initial problem of distinguishing it from any other scenery.

2. The setup and the main result

Before we give the first central result of our paper in this section let us quickly introduce and recall the most important notations. If not mentioned otherwise, in what follows ξ will always be a one-dimensional 3-color scenery, that is

$$\xi : \mathbb{Z} \to \{0, 1, 2\}.$$

Actually, the generalization to $m \geqslant 4$ is easy and straightforward, but there will also be examples with $m = 2$ where Theorem 2.1 applies. We will always choose ξ randomly from a class of sceneries in such a way that the conditions of Theorem 2.1 are fulfilled.

Moreover let $S = (S_k)_{k \in \mathbb{N}_0}$ be symmetric, simple random walk without holding on $\mathbb{Z}$ starting in the origin. The measure $\mathbb{P}$ will denote the product measure on the product space of all sceneries and all random walk paths (with the obvious marginals). Hence we will always assume independence of the walks and the sceneries. The central problem will be to reconstruct ξ from its color record

$$\chi := (\chi((S_k)_{k \in \mathbb{N}_0})) := ((\xi(S_k)_{k \in \mathbb{N}_0})$$

under S (by $\chi|[0, n]$ we will denote the first $n + 1$ observations). This means we want to find a measurable mapping

$$\mathscr{A} : \{0, 1, 2\}^{\mathbb{N}} \to \{0, 1, 2\}^{\mathbb{Z}}$$

such that $\mathbb{P}$-almost surely

$$\mathscr{A}(\chi) \sim \xi.$$

Here for two sceneries $\eta, \xi \in \{0,1,2\}^{\mathbb{Z}}$ we write $\xi \sim \eta$ (and say that they are equivalent), if there are $a \in \mathbb{Z}$ and $s \in \{-1,+1\}$ such that

$$\xi(z+a) = \eta(sz)$$

for all $z \in \mathbb{Z}$, i.e. ξ can be obtained from η by translation and reflection at the origin.

For the idea of the reconstruction algorithm the following representation of ξ as a path on a colored tree is essential. In our case this tree will be the 3-regular tree $T_3 := (V_3, E_3)$, that is the connected, unrooted, infinite tree with all its vertices $v \in V_3$ having degree 3. We choose one (arbitrary) vertex and call it the origin o. We color T_3 (or more precisely V_3) in three different ways φ^0, φ^1 and φ^2. Up to isomorphisms of the tree these colorings are uniquely defined by the color of the origin (or root) o

(i)

$$\varphi^i(o) = i$$

for $i = 0,1,2$ and the following construction rule

(ii) For each $v \in V_3$ let $\{v_1, v_2, v_3\}$ be the set of its neighboring vertices (according to the graph topology induced by E_3). Then

$$\{\varphi^i(v_1), \varphi^i(v_2), \varphi^i(v_3)\} = \{0,1,2\}$$

for each $i = 0,1,2$ and each $v \in V_3$.

Now we can represent ξ as a nearest neighbor path R on T_3 (taking the randomness in ξ into account this will be a random path, but note that other than in Matzinger, 1999a, R is not necessarily a random walk on T_3). This will be done in the following way.

(a) Choose the coloring φ^i with $i = \xi(0)$. (Note that we know $\xi(0)$ as the random walk S is supposed to start in zero).

(b) We let $R = (R(z))_{z \in \mathbb{Z}}$ be the nearest neighbor random path (that is $R(z)$ and $R(z+1)$ are adjacent for each $z \in \mathbb{Z}$) on T_3 colored with $\varphi^{\xi(0)}$ such that

$$R(0) = o$$

and

$$\varphi^{\xi(0)}(R(z)) = \xi(z)$$

for all $z \in \mathbb{Z}$.

Note that given T_3, the choice of o, and the coloring this path R is unique. This representation of ξ as a random path on T_3 colored with $\varphi^{\xi(0)}$ will indeed help us to reconstruct ξ up to equivalence. To this end note that knowing R plus knowing $\xi(0)$ is indeed equivalent to knowing ξ. Unfortunately, we do not know R but only $R \circ S$ (the latter because of

$$\varphi^{\xi(0)} \circ (R \circ S) = (\varphi^{\xi(0)} \circ R) \circ S = \xi \circ S = \chi$$

and thus we can reconstruct $R \circ S$ from χ).

Interestingly, the only knowledge we require about R in order to reconstruct ξ is that it is transient, that is that the random path R (again recall that R is random as ξ is random) visits each vertex $v \in V_3$ only finitely many times almost surely. This is a considerable improvement of Matzinger's previous result Matzinger, 1999a, who even for i.i.d. sceneries needed some further conditions. A major tool in the proof of the following theorem will consist of reformulation of this transience in terms of *crossings* of some pieces of the tree T_3 by R. For some $v \neq w \in V_3$ we say that a time interval $[s,t]$ (without loss of generality $s < t$—otherwise we just reverse the order of v and w) is a *crossing* of (v,w), if $R(s) = v$, $R(t) = w$, or $R(s) = w$, $R(t) = v$, and

$$R(s') \neq v \quad \text{and} \quad R(s') \neq w$$

for all $s < s' < t$. Observe that two crossings of (v,w) either agree or are disjoint (that is the time intervals are disjoint).

Moreover, we say that $[s,t]$ is a shortest crossing of (v,w) by R, if

$$t - s = \min\{|t' - s'|, [s',t'] \text{ is a crossing of } (v,w) \text{ by } R\}.$$

Now we are ready to formulate the first central result of this paper (the other will be stated in Section 5).

Theorem 2.1. *With the above definitions, assume that the random path R is transient. Then there exists a measurable mapping*

$$\mathscr{A} : \{0,1,2\}^{\mathbb{N}} \to \{0,1,2\}^{\mathbb{Z}}$$

such that

$$\mathbb{P}(\mathscr{A}(\chi) \sim \xi) = 1.$$

3. The algorithm

In this section, we are going to present the basic reconstruction scheme, while details will be left to the proofs to follow in the next section. The core of this algorithm consists of stopping the random walk S infinitely many times at the same place. We will see in the next section that this indeed is already enough to be able to reconstruct ξ up to equivalence. Actually this stopping of the random walk is of a different nature, when ξ is essentially symmetric (by this we mean that there is a finite interval $I = (a,b)$ such that $\xi(a - x) = \xi(x - b)$ for all $x \in \mathbb{N}$). Therefore, we first test ξ on essential symmetry. This symmetry can also be expressed in terms of the path R on T_3 which will exploit in the first step of the algorithm.

Step 1 (*Reconstruction Algorithm*): Test whether there are $v \neq w \in V_3$ such that there is only one shortest crossing of (v,w)

From Step 1 the algorithm proceeds in two different directions depending on whether it has been successful (that is there are v,w with only one shortest crossing of (v,w) by R) or not. The first case will be called Case A the other one Case B.

Case A: There is at least one pair $v \neq w$ such that there is only one shortest crossing of (v,w)

Here we proceed by producing infinitely many stopping times all stopping S at the same point.

Step 2 (*Reconstruction Algorithm*): Stop the random walk S infinitely often at the same point.

Finally, we use these stopping times to reconstruct ξ.

Step 3 (*Reconstruction Algorithm*): Reconstruct ξ up to equivalence with probability one from these stopping times.

In Case B, where for every $v \neq w \in V_3$ there are at least two shortest crossings of (v,w) by R, we have to apply a slightly different techniques.

Case B: For all $v,w \in V_3$ there are at least two shortest crossings of (v,w):

Step 2 (*Reconstruction Algorithm*): Stop the random walk S infinitely often at two different points.

Step 3 (*Reconstruction Algorithm*): Reconstruct ξ up to equivalence with probability one from these stopping times.

Of course, this is a very rough description of the algorithm. We will fill its different steps with life in the next section, where we prove that it actually works.

4. Proof that the algorithm works

In this section we show that under the condition that R is transient the algorithm actually reconstructs ξ up to equivalence with probability one. This proof is split into different parts. In the first part we show that if the walk is transient, then almost surely there are vertices $v,w \in V_3$ such that there are at most two crossings of (v,w) by R.

Definition 4.1. (1) Let W, W' be a set and $f : W \to W'$ be a mapping. Then $\operatorname{Im} f := \{f(w), w \in W\}$ denotes the image of f.

(2) Consider the 3-regular tree $T_3 := (V_3, E_3)$ and two vertices $v,w \in V_3$. The graph distance $d(v,w)$ is defined as minimum the minimum length of a path (minimum number of connected edges in E_3) to be crossed to get from v to w.

Lemma 4.2. *Let R be transient. Then for every fixed $v \in V_3$ and every sequence $v_n \in V_3$ in the image of R with $d(v,v_n) = n$ and $d(v_{n-1},v_n) = 1$ almost surely the number $N(n)$ of distinct shortest crossings of (v,v_n) by R converges and the limit is $\mathbb{P}$-almost surely either* 1 *or* 2.

Remark 4.3. Note that due to the transience of R the Image $\operatorname{Im} R$ of the representation of the scenery ξ on T_3 is almost surely infinite. Hence such a point v and a sequence of points v_n as assumed in the above lemma actually exist.

Proof of Lemma 4.2. Without loss of generality we will take v to be the origin o. By transience of R the origin is visited by R only finitely many times with probability one. Thus with probability one the random variables

$$t_{\max} := \max\{t \colon R(t) = 0\}$$

and

$$t_{\min} := \min\{t\colon\ R(t) = 0\}$$

are well defined and obey

$$-\infty < t_{\min} \leqslant 0 \leqslant t_{\max} < \infty.$$

In particular, $\Delta t := t_{\max} - t_{\min}$ is finite with probability one. Now take $v_n \in \operatorname{Im} R$ with $d(o, v_n) = n$ as assumed in the above lemma. Then for all n large enough $d(v_n, o) > \Delta t$. Moreover, since $v_n \in \operatorname{Im} R$ there exists $t_n \in \mathbb{Z}$ such that $R(t_n) = v_n$. As $d(v_n, o) > \Delta t$ we conclude that

$$t_n \notin [t_{\min}, t_{\max}].$$

Thus at most two crossings of (o, v_n) can occur, one of the form $(t_{\max}, t_1)$ and another one of the form $(t_2, t_{\min})$, where

$$t_1 := \min\{t > t_{\max}\colon\ R(t) = v_n\}$$

and

$$t_2 := \max\{t < t_{\min}\colon\ R(t) = v_n\}.$$

Also note that because $v_n \in \operatorname{Im} R$ one of the above two crossings really can be found. $\square$

As we will see in the next lemma not only we are either in Case A or in Case B, but there is also a test which reveals with probability one (given a full color record) in which of the two situations we are.

Lemma 4.4. *For each $v, w \in \operatorname{Im} R$ there exists a test that on the basis of the observations χ decides with probability one whether there is only one shortest crossing from v to w or whether there is more than one such shortest crossing.*

Proof. Let $v, w \in \operatorname{Im} R$ be any two points reached by the random walk R. Note that S as a random walk on the integers is recurrent and hence so is $R \circ S$ as a random path on T. Therefore, $R \circ S$ will pass every finite path in $\operatorname{Im} R$ infinitely often and thus,

$$\tilde{T} := \min\{|s - t|; R(S(s)) = v, R(S(t)) = w\}$$

estimates the time for the shortest crossing of (v, w) by R correctly with probability one.

Also note that the distribution function of the waiting time between two shortest crossings of (v, w) by $R \circ S$ is strictly larger if there is more than one shortest crossing of (v, w) by R than if there is just one such shortest crossing.

To be more specific, let W be the random variable that denotes the first time after which the random path $R \circ S$ has walked from v to w in $\tilde{T}$ steps:

$$W := \min\{n \geqslant 0, R \circ S(n - \tilde{T}) = v, R \circ S(n) = w\}.$$

Moreover, let F the distribution function of W conditioned on starting with S in the point z corresponding to the point w (via the representation R) of the unique shortest crossing of (v,w) by R at time 0, i.e.

$$F := \mathscr{L}(W \mid S(0)=z).$$

When there are several shortest crossings of (v,w) by R, say $[y_1,z_1],\dots,[y_k,z_k]$ with $k \geqslant 2$ is the set of all shortest crossings of (v,w) by R and (for our purposes without loss of generality) $R(z_1)=R(z_2)=\cdots=R(z_k)=w$ we denote with F_i the distribution of W when starting with S in the point w_i at time zero, $i=1,\dots,k$.

Note that the distribution function F then will be strictly smaller than each of the distribution functions F_i (i.e. $F(t) \leqslant F_i(t)$ for all t and that $F(t) < F_i(t)$ for all $t \geqslant T_0$ for some finite T_0). Indeed, if there are several shortest crossings of (v,w) by R, then a crossing from v to w by $R \circ S$ in $\tilde{T}$ steps may be caused by crossing one of several intervals $[y_1,z_1],\dots,[y_k,z_k]$ in $\tilde{T}$ steps. Now the event to cross one of $[y_1,z_1],\dots,[y_k,z_k]$ in $\tilde{T}$ steps has a higher probability than to cross a fixed interval $[y,z]$ in $\tilde{T}$ steps in case there is just one shortest crossing of (v,w) by R. This will eventually also show up in the distribution function of W.

Moreover, notice that F can be explicitly calculated when we know that there is only one shortest crossing from v to w and we also know its length. Indeed, denoting by l the length of such a shortest crossing and considering the renewal process, with a renewal after every time where the random path $R \circ S$ has walked from v to w in l steps, we can calculate the probability that a time t is a renewal time. As a matter of fact, this can be done by observing that if there is only one shortest crossing of (v,w), this crossing corresponds (by the random path R) to two points $z_1,z_2 \in \mathbb{Z}$ with $R(z_1)=v$ and $R(z_2)=w$ and $|z_1-z_2|=l$. So the probability of having a renewal at time t equals the probability of walking with S from z_2 to z_1 in $t-l$ steps times 2^{-l} (for a straight crossing from z_1 to z_2). By a standard exercise in renewal theory, this also yields the probability that t is the time of a first renewal, hence the distribution function of W.

As we can also estimate the length l by $\tilde{T}$ with probability one correctly, we can, in principal, calculate the distribution function of W in the case where there is only one shortest crossing of (v,w) by R.

Finally, we can also test whether there is only one shortest crossing of (v,w) by R correctly with probability one.

In fact, if $[s_1,t_1],[s_2,t_2],\dots$ denotes all intervals where the random path $R \circ S$ walks from v to w in $l=\tilde{T}$ steps, so $s_1 < t_1 < s_2 < t_2 < \cdots$ and $R(S(s_i))=v$ and $R(S(t_i))=w$ for all $i=1,2,\dots$. Then by the law of large numbers (Glivenko–Cantelli lemma) the empirical distribution function of the "first renewal times"

$$\frac{1}{n-1}\sum_{i=2}^{n} \delta_{t_i-t_{i-1}}$$

converges to some distribution function $\bar{F}$ with probability one as n goes to infinity (of course, again, here we exploit the recurrence of S which gives us infinitely many such crossings).

If there is only one shortest crossing of (v,w) by R, the distribution function $\bar{F}$ will equal F with probability one, otherwise $\bar{F}$ will be a mixture of the F_i, hence larger than F.

So, if

$$\lim_{n\to\infty}\frac{1}{n-1}\sum_{i=2}^{n}\delta_{t_i-t_{i-1}}$$

differs from F (which we can calculate as indicated above) we conclude that there is more than one such shortest crossing, otherwise we decide that there is only one shortest crossing of (v,w) by R. As has been shown above this test succeeds in giving the correct number of shortest crossings of (v,w) with probability one. $\square$

In the next steps we will see that the algorithm actually works in Case A. To this end we first have to show that we can indeed stop the random walk S infinitely often at the same place. So, let

$$\mathscr{H}_k := \sigma\{\xi(z), z\in\mathbb{Z}, S(0),\ldots,S(k)\}$$

and define the filtration $\mathscr{H}$ as

$$\mathscr{H} := \{\mathscr{H}_k, k\in\mathbb{N}\}.$$

Lemma 4.5. *If there are $v,w\in\operatorname{Im}R$ such that there is only one shortest crossing of (v,w) by R we can stop the random walk infinitely often at the same place, i.e. we are able to construct an infinite sequence of increasing stopping times $\tau(1),\tau(2),\ldots$ with respect to the filtration $\mathscr{H}$ such that*

$$S(\tau(1)) = S(\tau(2)) = \ldots = S(\tau(k)) = \ldots\ .$$

Remark 4.6. Observe that as has been already discussed in the context of Lemma 4.2 and in particular when motivating the algorithm in Section 3, Case A is the relevant case for most distributions of the scenery we might think of. Indeed, whenever the distribution of the scenery exhibits some form of asymptotic independence, for example, the scenery will a.s. not be essentially symmetric and thus we will almost surely be in Case A.

Proof of Lemma 4.5. Let $v,w\in\operatorname{Im}R$ such that there is only one shortest crossing of (v,w) by R. Let the length of this shortest crossing be $\tilde{L}$. This length $\tilde{L}$ can be estimated correctly with probability one by

$$\tilde{T} = \min\{|s-t|; R(S(s)) = v, R(S(t)) = w\}.$$

Thus, whenever we observe that the random walk $R\circ S$ (which can reconstruct from χ) walks from v to w, in time $\tilde{T}$ we know that also with probability one the random walk S must be at the same place when $R\circ S$ has reached w. Hence we can construct a stopping rule and stop S, whenever $R\circ S$ has walked from v to w in time $\tilde{T}$. This rule stops S always at the same place. Now, by recurrence of S with probability one there are infinitely many time intervals of length $\tilde{T}$ where $R\circ S$ walks from v to w, and thus the above rule stops S infinitely often at the same place with probability one.

At first glance it might seem that the sequence of stopping times $\tau(1), \tau(2), \ldots$ thus obtained is not $\mathscr{H}$-adapted in the above sense, since their definition involves $\tilde{T}$ which only is measurable with respect to the whole path $(S(t))_{t\in\mathbb{N}}$. On the other hand, given the scenery ξ, that is in particular given $\tilde{L}$, we are able to construct stopping times $\tau'(1), \tau'(2), \ldots$, such that $\tau'(k)$ stops the random walk $R \circ S$ when it has walked from v to w in $\tilde{L}$ steps for the kth time. Obviously the $\tau'(k)$ are $\mathscr{H}$-adapted in the above sense. On the other hand, the sequences $\tau(1), \tau(2), \ldots$ and $\tau'(1), \tau'(2), \ldots$ are equal $\mathbb{P}$-almost surely. It follows that the sequence $\tau(1), \tau(2), \ldots$ is $\mathscr{H}$-adapted. $\square$

Finally, we shall see that a rule that stops S infinitely often at the same place actually is helpful to reconstruct ξ.

Lemma 4.7. *If we can create a stopping rule (that is an infinite sequence of $\mathscr{H}$-adapted stopping times) that stops S infinitely often at the same place, we can also reconstruct ξ restricted to any finite interval (up to equivalence) with probability one.*

Proof. We will prove this lemma by induction.

Say, we stop the random walk infinitely often in the point $z \in \mathbb{Z}$. Of course, we then know the color of z. To find out the color of $z-1$ and $z+1$ we let the random walk S run one further step (after we have stopped it in z) and read off the color of the next point. As we have infinitely many such stopping times we will eventually see both, the color of $z+1$ and the color of $z-1$ with probability one. Since we are only interested in reconstruction up to shifts and reflection of the scenery this knowledge suffices to reconstruct ξ on $[z-1, z+1]$. This is the beginning of the induction.

For the induction step assume we already have reconstructed ξ up to shifts and reflection on the interval $[z-n, z+n]$. First assume that ξ is not symmetric under reflection at z on $[z-n, z+n]$, that is $(\xi(z-1), \ldots, \xi(z-n)) \neq (\xi(z+1), \ldots, \xi(z+n))$. The other case will be treated similarly at the end of this proof.

First we introduce the set of all nearest neighbor paths of length $n+1$ that starting in z in the first n steps read the same color sequence as a straight walk to the right:

$$\mathscr{S}_n := \{\rho\colon \{0, \ldots, n+1\} \to \mathbb{Z}\colon \rho(0) = z, |\rho(t+1) - \rho(t)| = 1,\ \forall t = 0, \ldots, n$$
$$\text{and} \quad \xi(\rho(t)) = \xi(z+t)\ \forall t = 0, \ldots, n\}$$

and its subset where we exclude the straight walk to the right (the straight walk to the left is automatically excluded as we have already assumed that ξ is non-symmetric with respect to reflection at z)

$$\mathscr{S}'_n := \{\rho \in \mathscr{S}_n\colon \rho(n) \neq z+n\}.$$

With the help of the sets $\mathscr{S}_n$ and $\mathscr{S}'_n$ we construct two measures on the space $\{0, 1, 2\}$:

$$\pi_n(\cdot) := \frac{1}{|\mathscr{S}_n|} \sum_{\rho \in \mathscr{S}_n} \delta_{\xi(\rho(n+1))}(\cdot)$$

and

$$\pi'_n(\cdot) := \frac{1}{|\mathscr{S}'_n|} \sum_{\rho \in \mathscr{S}'_n} \delta_{\xi(\rho(n+1))}(\cdot)$$

(if $\mathscr{S}'_n = \emptyset$ we simply set $\pi'_n \equiv 0$.)

Now the following three observations are crucial: First note that the desired color of $z+n+1$ is the only color with a higher value (probability) under π_n than under π'_n, hence

$$\xi(z+n+1) = \operatorname{supp}((\pi_n(\cdot) - \pi'_n(\cdot))^+),$$

where supp denotes the support of a function and for a real number a we write a^+ for $\sup\{a, 0\}$.

Second, observe that from knowing $\xi|[z-n, z+n]$ (which we know by our induction hypotheses), we can construct $\mathscr{S}'_n$ and therefore also calculate π'_n.

Finally, we also have an arbitrarily good approximation for π_n. Indeed, let us denote by ϑ^T the set of all times $t \leqslant T$ where we stop the random walk S in the point z and read the colors $\xi(z+i)$ in the next n steps. More precisely

$$\vartheta^T := \{t \leqslant T\colon\ S(t) = z \text{ and } \xi(S(t+i)) = \xi(z+i), i = 0, \ldots, n\}.$$

Then by the strong Markov property of the stopping times and the law of large numbers

$$\tilde{\pi}_n^T(\cdot) := \frac{1}{|\vartheta^T|} \sum_{t \in \vartheta^T} \delta_{\xi(S(t+n+1))}(\cdot)$$

converges to π_n, when T becomes large. Thus with probability one

$$\lim_{T\to\infty} \operatorname{supp}((\pi_n^T - \pi'_n)^+)$$

consists of precisely one element and reveals the color of $z+n+1$.

The same technique can be applied to reconstruct $\xi(z-n-1)$. To this end we simply replace $\mathscr{S}_n$ by $\bar{\mathscr{S}}_n$ defined as

$$\bar{\mathscr{S}}_n := \{\rho\colon\ \{0, \ldots, n+1\} \to \mathbb{Z} \mid \rho(0) = z, |\rho(t+1) - \rho(t)| = 1,\ \forall t = 0, \ldots, n$$
$$\text{and } \xi(\rho(t)) = \xi(z-t)\ \forall t = 0, \ldots, n\}$$

and $\mathscr{S}'_n$ by

$$\bar{\mathscr{S}}'_n := \{\rho \in \bar{\mathscr{S}}_n\colon\ \rho(n) \neq z-n\}$$

and proceed as above.

If finally, ξ restricted to $[z-n, z+n]$ is symmetric under reflection at z, the support of $(\pi_n - \pi'_n)^+$ (π_n and π'_n defined as above) may either consist of one or two elements. More precisely, it will be one elementary, if also $\xi(z-n-1) = \xi(z+n+1)$, in which case we simply assign this color to each of the two vertices $z-n-1$ and $z+n+1$. If $\xi(z-n-1) \neq \xi(z+n+1)$ indeed

$$\operatorname{supp}((\pi_n(\cdot) - \pi'_n(\cdot))^+)$$

and also

$$\lim_{T\to\infty} \operatorname{supp}((\pi_n^T - \pi'_n)^+)$$

will consist of two elements (with the notation introduced above the latter will be the "right colors" with probability one, again). As we only aim at reconstructing ξ up to translations and reflections we do not to need to care about to which of $z-n-1$ and $z+n+1$ we assign which of the two colors.

This finishes the proof of the lemma. □

Finally, we remark that Lemma 4.7 implies that we can reconstruct ξ up to equivalence with probability one

Corollary 4.8. *If we can create a stopping rule* (*that is an infinite sequence of* $\mathscr{H}$*-adapted stopping times*) *that stops* S *infinitely often at the same place*, *we can also find a mapping* $\mathscr{A}:\{0,1,2\}^{\mathbb{N}}\to\{0,1,2\}^{\mathbb{Z}}$ *with*

$$\mathscr{A}(\chi)\sim\xi.$$

Proof. Just paste the different pieces of scenery together. □

So the above steps show that the algorithm proposed in Section 3 works in Case A. The next lemmata will show that the same holds true in Case B.

To this end one strategy would be to give the equivalent of Lemma 4.4 in the sense given; we know that for each $v,w\in \operatorname{Im} R$ there exists at least two shortest crossings of (v,w) by R, and we want to test whether there are precisely two such crossings or if there are more than two. Such a test may be very difficult to find. Indeed, recall that every shortest crossing of (v,w) by R corresponds to two points $z_1,z_2\in\mathbb{Z}$ with $R(z_1)=v$ and $R(z_2)=w$. Now it may be very hard to decide at first glance from the empirical distribution function of the waiting time between two shortest walks from v to w by $R\circ S$ whether we have two such intervals which are far apart from each other or whether we have three (or more) of them which are rather close.

To overcome this difficulty we apply another strategy. We first demonstrate how to reconstruct ξ (up to equivalence) if we know that there are $v,w\in V_3$ for which there are precisely two shortest crossings of (v,w) by R. We then see in a final step that in view of Lemma 4.11 this technique already suffices to find a general reconstruction algorithm.

The situation where for each pair $v,w\in \operatorname{Im} R$ there at least two shortest crossings of (v,w) by R can again be split into two different cases. In the first case there are $v,w\in \operatorname{Im} R$ such that there is a shortest crossing of (v,w) by R with a color sequence different of all other shortest crossing. By this we mean that there is a shortest crossing of (v,w) by R, say the first shortest crossing, such that the sequence of colors read by R when going from v to w say, is different from the corresponding sequence of colors for all other shortest crossing of (v,w) by R. We will call such shortest crossings *distinct* as opposed to the *non-distinct* shortest crossings, where each sequence of colors read by R when following such a shortest crossing of (v,w) agrees with the sequence of colors of another shortest crossing of (v,w).

The case of distinct shortest crossings is quite similar to Case A, and we will also apply Lemma 4.7 for the reconstruction.

Before doing so we show that we really can find out whether there are distinct or non-distinct shortest crossings of (v,w) by R.

Lemma 4.9. *Assume that for each $v, w \in V_3$ there are at least two shortest crossings of (v, w) by R, then there is a test which decides with probability one whether these crossings are distinct or not.*

Proof. Note that due to the recurrence of S (and hence of $R \circ S$) the random path $R \circ S$ will follow each shortest crossing of (v, w) by R infinitely often. These "direct" passages from v to w can be determined as above, since they are the only ones happening in the "shortest observed time"

$$\tilde{T} = \min\{|s - t|; R(S(s)) = v, R(S(t)) = w\}$$

with probability one. So comparing the color record (that is the color read during such a fastest passage) of these shortest crossings from v to w, we see whether there is only one such color record or whether there are different ones. In the first case the shortest crossings are definitely non-distinct while in the latter case we can test whether the limiting empirical distribution function of any of these fixed color records is different from the distribution function of a color record of length $\tilde{T}$, given that it is produced on one shortest crossing between two points at distance $\tilde{T}$. Exactly as in the proof of Lemma 4.5 we conclude that the crossings are distinct if the two distribution functions above agree, otherwise we deduce that the crossings are non-distinct. As in Lemma 4.5 this test works with probability one. □

Next we see that we can reconstruct ξ if we can find (v, w) with distinct shortest crossings of (v, w) by R.

Lemma 4.10. *Assume that there are $v, w \in V_3$ such that there are at least two shortest crossings of (v, w) by R and assume that these shortest crossings are distinct. Then we can find a mapping $\mathscr{A} : \{0, 1, 2\}^{\mathbb{N}} \to \{0, 1, 2\}^{\mathbb{Z}}$ with*

$$\mathscr{A}(\chi) \sim \xi.$$

(*Note that in this case $\mathscr{A}$ will depend in general on R and (v, w).*)

Proof. Recall that we call shortest crossings of (v, w) by R distinct if there is one shortest crossing, say $[s, t]$, of (v, w) by R such that the sequence of colors read by R when going from v to w say, in time $[s, t]$, is different from the corresponding sequence of colors for all other shortest crossing of (v, w) by R. Also note that by the representation of ξ as a random path R on T_3 there is an unique interval $[z_1, z_2] \subset \mathbb{Z}$ corresponding to this unique shortest crossing $[s, t]$ of (v, w) by R (without loss of generality $R(z_2) = w$).

Hence, whenever the random path $R \circ S$ walks from v to w in time $\tilde{T}$ and produces the color record characteristic for the unique shortest crossing $[s, t]$ of (v, w) by R we know that the random walk S is in a certain point, namely that it is in z_2. By recurrence of S this will happen infinitely often with probability one, thus we have a stopping rule which allows us to stop S infinitely often at the same point z_2 with probability one. Thus we can apply Lemma 4.7 together with Corollary 4.8 to prove the statement of the lemma. □

Next we will see what to do, if we know that there are $v, w \in V_3$ for which there are exactly two non-distinct crossings.

Lemma 4.11. *Assume that there exists $v, w \in V_3$ for which there are precisely two crossings of (v, w) by R. Then we can find a mapping $\mathscr{A} : \{0,1,2\}^{\mathbb{N}} \to \{0,1,2\}^{\mathbb{Z}}$ with*

$$\mathscr{A}(\chi) \sim \xi$$

for $\mathbb{P}$-almost all walks S and fixed R (note that $\mathscr{A}$ will depend on v and w).

Proof. Note that we only need to prove this theorem in the case where the two crossings have the same length τ and are non-distinct, otherwise there is one shortest crossing or Lemma 4.10 applies.

Again we will prove this lemma in two steps. In the first step we will show how to reconstruct every finite piece of scenery under the assumptions of the lemma. In the second (short) part we then will prove that this already suffices to reconstruct the whole scenery (up to equivalence).

So let us assume we know that for two fixed points $v \neq w \in V_3$ there are precisely two crossings of (v, w) by R. By the representation R of ξ these two crossings correspond to two intervals, say $[z_1, z_2]$ and $[z_1', z_2']$ (without loss of generality $z_2 < z_1'$ with

$$|z_2 - z_1| = |z_2' - z_1'| = \tau,$$

$R(z_1) = R(z_2') = v$, $R(z_2) = R(z_1') = w$ or $R(z_1) = R(z_2') = w$, $R(z_2) = R(z_1') = v$ (other situations for the z_1, z_2, z_1', z_2' cannot occur due to our assumption that there are only two crossings of (v, w) by R), and such that $\xi(z_1 + x) = \xi(z_2' - x)$ for all $0 \leqslant x \leqslant \tau$.

Now, first of all note that we can estimate τ by

$$\tilde{T} = \min\{|s - t|; R(S(s)) = v, R(S(t)) = w\}$$

(and this estimate is correct with probability one).

Moreover, we can also give an accurate estimate for $z_1' - z_2$. Indeed, observe that the empirical distribution function of the observed walks from v to w converges. More precisely let $[s_1, t_1], [s_2, t_2], \ldots$ denote all intervals where the random path $R \circ S$ walks from v to w in $\tau = \tilde{T}$ steps, so $s_1 < t_1 < s_2 < t_2 < \cdots$ and $R(S(s_i)) = v$ and $R(S(t_i)) = w$ for all $i = 1, 2, \ldots$. Then by the law of large numbers (Glivenko–Cantelli Lemma) the empirical distribution function of the "first renewal times"

$$\frac{1}{n-1} \sum_{i=2}^{n} \delta_{t_i - t_{i-1}}$$

converges to some distribution function $\bar{F}$ with probability one as n goes to infinity. Then with probability one $\bar{F}$ will be different from the distribution function F we would have, if there were only one crossing from v to w. Note also, as already remarked in the proof of Lemma 4.4 that we can actually calculate F. Denote by θ_1 the smallest t where F and $\bar{F}$ differ, that is

$$\theta := \inf\{t : F(t) \neq \bar{F}(t)\}.$$

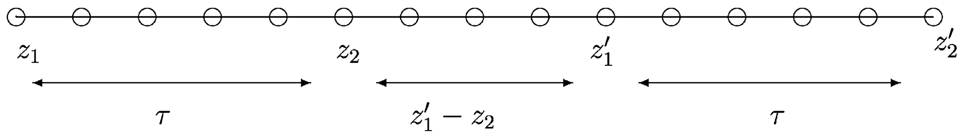

Fig. 1. Intervals for the shortest crossings.

As illustrated in Fig. 1 below (where for a moment we assume that $R(z_1) = R(z_2') = w$, $R(z_2) = R(z_1') = v$) the first possible t for which $F(t)$ and $\bar{F}(t)$ can differ is exactly $z_1' - z_2 + 2\tau$. This is true because for $t = z_1' - z_2 + 2\tau$ the walk, instead of walking from z_1' to z_2' in τ steps then back again to z_1' in another τ steps to create another short crossing of (w, v) from there (which amounts in a renewal time of 2τ), may walk directly from z_1' over z_2' to z_2 and create a shortest crossing of (w, v) from there. Also, if the shortest crossing is from v to w in $t = z_1' - z_2 + 2\tau$ steps the walk may e.g. walk from z_1 to z_2' directly and create a shortest crossing of (v, w) from z_2. As these events have a positive probability to occur after $t = z_1' - z_2 + 2\tau$ steps, this shows that

$$\theta := \inf\{t \colon F(t) \neq \bar{F}(t)\}$$

is a good estimator of $z_1' - z_2 + 2\tau$. Moreover, $\tilde{T}$ estimates τ with probability one correctly, thus with probability one

$$\theta - 2\tilde{T} = z_1' - z_2.$$

Moreover, we can also deduce in which of the two situations $R(z_1) = R(z_2') = v$, $R(z_2) = R(z_1') = w$ or $R(z_1) = R(z_2') = w$, $R(z_2) = R(z_1') = v$ we are.

Indeed, with probability one: if $R(z_1) = R(z_2') = v$ and $R(z_2) = R(z_1') = w$ then the only way to observe a crossing from w to v created on the interval $[z_1, z_2]$ exactly θ steps after a crossing from w to v created on the interval $[z_1', z_2']$ is by observing a shortest crossing from v to w in time steps $\tau + 1$ to 2τ. On the other hand, one might well observe a crossing from v to w created on the interval $[z_1, z_2]$ exactly θ steps after a crossing from v to w created on the interval $[z_1', z_2']$ is without a shortest crossing from w to v in time steps $\tau + 1$ to 2τ. Thus in the case that $R(z_1) = R(z_2') = v$ and $R(z_2) = R(z_1') = w$ we have that

$$\mathbb{P}(G_{wv}^{\theta}) < \mathbb{P}(G_{vw}^{\theta}).$$

Here

$$G_{wv}^{\theta} := \{\text{There are shortest crossing from } w \text{ to } v \text{ which are exactly } \theta \text{ steps apart}$$
$$\text{and there is } \textit{no} \text{ shortest crossing from } v \text{ to } w \text{ in steps } \tau + 1 \text{ to } 2\tau\}$$

and

$$G_{vw}^{\theta} := \{\text{There are shortest crossing from } v \text{ to } w \text{ which are exactly } \theta \text{ steps apart}$$
$$\text{and there is } \textit{no} \text{ shortest crossing from } w \text{ to } v \text{ in steps } \tau + 1 \text{ to } 2\tau\}.$$

If on the other hand $R(z_1) = R(z_2') = w$ and $R(z_2) = R(z_1') = v$, then we have

$$\mathbb{P}(G_{wv}^{\theta}) > \mathbb{P}(G_{vw}^{\theta}).$$

Now the probabilities $\mathbb{P}(G^{\theta}_{wv})$ and $\mathbb{P}(G^{\theta}_{vw})$ can be arbitrarily well approximated by their corresponding empirical probabilities. Hence we have a test that decides with probability one correctly in which of the two situations $R(z_1)=R(z_2')=v$, $R(z_2)=R(z_1')=w$ or $R(z_1)=R(z_2')=w$, $R(z_2)=R(z_1')=v$ we are. For the rest of this proof we will without loss of generality assume that $R(z_1)=R(z_2')=w$, $R(z_2)=R(z_1')=v$.

Next we will reconstruct $\xi|[z_2,z_1']$. To this end we stop the random walk S, whenever $R\circ S$ has walked from w to v in time $\tilde{T}$ (which will happen infinitely often with probability one, again, since S is recurrent). According to the above we then know with probability one that S is either in z_2 or in z_1', both with positive probability. Just as in the proof of Lemma 4.7 the empirical distribution of the first color read after $R\circ S$ has passed from w to v in time $\tilde{T}$ then reveals the color of the points neighboring v but outside $[z_1,z_2]$ and $[z_1',z_2']$.

To understand this in greater detail, first consider the Markov chain on $\{z_2,z_1'\}$ that enters z_2 after each time S has walked from z_1 to z_2 in time τ and that enters z_1' after each time S has walked from z_2' to z_1' in time τ and otherwise stays where it is. It is easy to see that this Markov chain has the uniform distribution (charging each of z_2', z_1' with probability $\frac{1}{2}$) as its invariant measure. Hence also, the empirical distribution of the colors read in the next step after each time the path $R\circ S$ has walked from w to v will converge to a distribution that assigns probability $\frac{1}{4}$ to each of $\xi(z_2-1)$, $\xi(z_2+1)$, $\xi(z_1'-1)$, and $\xi(z_1'+1)$ (of course, some of these colors will agree, in this case the probability of this color is just the sum of the above probabilities). Now note that we indeed know $\xi(z_2-1)=\xi(z_1'+1)$. Hence we are able to figure out the colors of $\xi(z_2+1)$ and $\xi(z_1'-1)$. As a matter of fact, if the limiting empirical distribution π of the colors read in the next step after each time the path $R\circ S$ has walked from w to v, satisfies $\pi(\xi(z_2-1))=1$, then

$$\xi(z_2+1)=\xi(z_1'-1)=\xi(z_2-1)=\xi(z_1'+1).$$

If $\pi(\xi(z_2-1))=\frac{3}{4}$, then there will be exactly one $i\in\{0,1,2\}\setminus\{\xi(z_2-1)\}$ with $\pi(i)\neq 0$ and the colors of $\xi(z_2+1)$ and $\xi(z_1'-1)$ will be this i and $\xi(z_2-1)$. If finally $\pi(\xi(z_2-1))=\frac{1}{2}$ there will be one or two i's with $i\in\{0,1,2\}\setminus\{\xi(z_2-1)\}$ and $\pi(i)\neq 0$ the color(s) of $\xi(z_2+1)$ and $\xi(z_1'-1)$ will be this i (resp. these i's). Actually, $\pi(\xi(z_2-1))$ cannot be less that $\frac{1}{2}$ since $\xi(z_2-1)=\xi(z_1'-1)$.

Following these ideas and the ideas already presented in the proof of Lemma 4.7 we are then able to reconstruct ξ inductively on $[z_2,z_1']$ (note that due to symmetry we do not need to care about whether we reconstructed ξ from z_2 to z_1' or from z_1' to z_2).

So up to now, we know $\xi|[z_1,z_2']$ (up to reflection symmetry). It remains to reconstruct ξ on $\mathbb{Z}\setminus[z_1,z_2']$. To this end recall that we are in the situation where between every two points $v_1\neq v_2\in\operatorname{Im}R$ there are at least two shortest crossings of (v_1,v_2) by R and that there are exactly two crossings of (v,w) by R. Now let us take a sequence of vertices $(v_n)_{n\in\mathbb{N}_0}$ in V_3 such that $v_0=w$, $d(v_n,v_{n+1})=1$, and such that $d(v_n,v)$ increases. Note that then there can be at most two crossings of (v_n,v) by R. According to the above we can then reconstruct $\xi|[z_1^{(n)},z_2^{(n)}]$, where $z_1^{(n)},z_2^{(n)}\in\mathbb{Z}$ are associated to v_n via the representation of ξ as a random path on T_3 (more precisely the intervals $[z_1^{(n)},z_2]$ and $[z_1',z_2^{(n)}]$ are the intervals associated to the crossings of (v,v_n) by R). As

$d(v,v_n)\to\infty$ also $|z_1^{(n)}|\to\infty$ and $|z_2^{(n)}|\to\infty$. Hence we can find an algorithm that reconstructs $\xi|[a,b]$ for each finite interval $[a,b]$.

The last step is just as in Corollary 4.8 to concatenate the different reconstruction to reconstruct ξ up to equivalence on all of $\mathbb{Z}$. □

The last step in proving that the algorithm proposed in Section 3 really works consists of showing that Lemma 4.11 implies that it works in Case B.

Lemma 4.12. *In Case B we can find a mapping* $\mathscr{A}:\{0,1,2\}^{\mathbb{N}}\to\{0,1,2\}^{\mathbb{Z}}$ *with*

$$\mathscr{A}(\chi)\sim\xi.$$

Proof. All what is left to show is how we can get into the situation of Lemma 4.11, in particular, how we can guarantee the existence of two points $v,w\in V_3$ such that there are precisely two crossings of (v,w) by R.

To this end take any sequence $v_n\in\operatorname{Im}R$ such that $d(v_n,v_{n+1})=1$ and $d(o,v_n)$ tends to infinity (o the origin of T_3). Then, according to Lemma 4.2, the number of crossings N_n of (o,v_n) by R with probability one converges to a limit which is either 1 or 2. As we are in Case B this limit can only be 2. As N_n is always an integer this means that N_n equals 2 for all but a finitely number of n's. Hence if we apply the reconstruction algorithm proposed in Lemma 4.11 to $v=o$ and $w=v_n$, we will get the correct scenery with probability one for all but a finite number of n's. Thus, if we denote by $\mathscr{A}^n$ the reconstruction proposed in Lemma 4.11 based on the points $v=o$ and $w=v_n$, we know that

$$\mathscr{A}(\chi):=\lim_{n\to\infty}\mathscr{A}^n(\chi)$$

exists with probability one (up to equivalence) in any reasonable topology as the sequence will be essentially constant (constant for all but a finite number of n's). With probability one this limit will agree (up to equivalence) with ξ. □

This finishes the proof of the fact that the algorithm proposed in Section 3 works.

5. Examples

In this section we shall discuss situations where the only assumption of Theorem 2.1, the transience of the representation R of the scenery ξ is satisfied and also such examples, where this assumption is violated, although the distribution of the colors is stationary and ergodic.

Before we start with these examples, let us remark that, of course, the situation where the colors are the output of an i.i.d. experiment, that is the situation where $\xi(z)$ are i.i.d. random variables for $z\in\mathbb{Z}$ and

$$\min\{\mathbb{P}(\xi(0)=0),\mathbb{P}(\xi(0)=1),\mathbb{P}(\xi(0)=2)\}>0$$

is covered by Theorem 2.1. As a matter of fact, this was already shown by one of the authors in Matzinger (1999a) and has been the starting point of the present paper.

Before presenting a large class of examples where the condition of the transience of R is satisfied, we first discuss three counterexamples, which will motivate the conditions in this main class of examples given in Theorem 5.9 below. The counterexamples will show that in a certain sense the class of distribution we give below is the largest "natural class" for Theorem 2.1 to hold.

The first example will be one, where R trivially cannot be transient.

Example 5.1. Consider a distribution of ξ produced by the following mechanism: Take a time-homogeneous Markov chain X_n on the set of colors $\{0,1,2\}$ with the following transition probabilities:

$$P(X_{n+1}=0|X_n=0)=P(X_{n+1}=1|X_n=0)=P(X_{n+1}=2|X_n=0)=\tfrac{1}{3}$$

and

$$P(X_{n+1}=0|X_n=1)=P(X_{n+1}=0|X_n=2)=1.$$

This Markov chain is irreducible and aperiodic (due to the holding in 0), and hence ergodic (even mixing of any kind). Now choose a coloring of the integers $\mathbb{Z}$ according to X_n, by, for example, attaching the color 0 to $0\in\mathbb{Z}$, and then first coloring the positive integers $\mathbb{Z}_+$ according to X_n and then Z_- independently according to the same distribution. Then, of course, the distribution of the colors inherits the properties of the Markov chain X_n, in particular, it admits a stationary distribution, is ergodic and mixing of any kind.

Note, however, that $\operatorname{Im} R$ consists of 6 points only. Thus, of course, R as a random path on T_3, cannot be transient and hence the main assumption of Theorem 2.1 is not fulfilled.

The above example illustrated that R might be recurrent, although all nice ergodic properties (such as stationarity and mixing properties) are fulfilled. The reason, of course, is that, as demonstrated above, R despite of fulfilling all these nice properties, may still have a finite image. The next example shows that on the other hand also, $\operatorname{Im} R$ may be infinite and R is still not transient.

Example 5.2. Probably the easiest example where $\operatorname{Im} R$ is infinite and still not transient, is that of a one-dimensional walk. More precisely, we choose

$$\mathbb{P}(\xi(0)=0)=\mathbb{P}(\xi(0)=1)=1/2,$$

and, of course, $\mathbb{P}(\xi(0)=2)=0$ and let the $\xi(z), z\in\mathbb{Z}$ be i.i.d. Then R is equivalent to a one-dimensional random walk on the integers $\mathbb{Z}$ without drift and holding. As, of course, such a random walk is recurrent, so is R and hence the main assumption of Theorem 2.1 is again not fulfilled.

This example might, of course, not be too surprising, as it is well-known that a one-dimensional random walk on the integers $\mathbb{Z}$ without drift and holding is recurrent. However, we gave this example, as it is the building block of the following example, which is definitely more surprising. It basically states that the condition of transience of R might even be violated, when $\operatorname{Im} R$ is infinite and the distribution of the colors has nice ergodic properties. Even more is true: in the example below $\operatorname{Im} R$ will be as "truly two-dimensional" as possible, in the sense that there are three infinite branches in $\operatorname{Im} R$.

Example 5.3. Consider the distribution of ξ produced by the following random mechanism. Take the following set of words (by which we mean a sequence of colors)

$$\mathscr{S} := \bigcup_{l \geqslant 0} \{(x_0, x_1, \dots, x_{2l}) :$$

$$x_0 \in \{0,1\}, x_1, \dots, x_l \in \{0,1,2\}, x_{l+x} = x_{l-x}, x = 1, \dots l\}$$

and introduce the following probability distribution π on $\mathscr{S}$:

$$\pi((x_0, x_1, \dots, x_{2l})) = \frac{1}{2^{l+2} 3^l}$$

for a $(x_0, x_1, \dots, x_{2l}) \in \mathscr{S}$.

Moreover, let us choose a random scenery ξ according to π in the following way. We choose two independent sequences of independent words according to π. Moreover, with probability $\frac{1}{2}$ we choose the starting point of the scenery to be either 0 or 1. If the starting point is 1 we attach the first of the two random sequences to the positive integers starting with 1, that is we attach the first word of, say L letters, to $1, \dots, L \in \mathbb{N}$, after that the next word of, say L' letters, to the points $L+1, \dots, L+L'$, and so on. After that we do the same thing for the second sequence and the non-positive integers $\mathbb{Z}_0^-$. If the starting point is 0, we attach the first sequence in the same way to $\mathbb{Z}_0^+$ and the second sequence to $\mathbb{Z}^-$.

Discussion of Example 5.3. First note that π is indeed a probability distribution on $\mathscr{S}$. Indeed, selecting an element from $\mathscr{S}$ according to π corresponds to first choosing its length $2l+1$ (note that $\mathscr{S}$ only consists of vectors of an odd length) with probability $2^{-(l+1)}$ (which works as $\sum_{l \geqslant 0} 2^{-l-1} = 1$) and then selecting one of the elements of $\mathscr{S}$ of length $2l+1$ with uniform probability. Note that, as a matter of fact, there are 2×3^l different choices for $(x_0, x_1, \dots, x_{2l}) \in \mathscr{S}$. Hence π is indeed a probability on $\mathscr{S}$.

With the help of renewal theory we now show that the sequence of colors produced by this mechanism is stationary and ergodic. Indeed, if we consider the renewal process, such that there is a renewal time, whenever a word from $\mathscr{S}$ is finished, then the greatest common divisor of these renewal times is one (since all words have odd length) and the mean renewal time is finite (which follows immediately from the definition of π). Hence it follows from renewal theory that there exists a stationary measure for the renewal times. Hence also the corresponding distribution of the colors on $\mathbb{Z}$ inherits this stationarity property.

To see that this distribution also is mixing and hence ergodic we have to understand that the shift is ergodic under the distribution induced by π. So let Θ be the right shift on $\mathbb{Z}$. We have to prove that for any two measurable events A and B

$$\lim_{t \to \infty} \mathbb{P}(\Theta^t(B) | A) \to \mathbb{P}(B).$$

First of all observe that for every $A, B \in \sigma(\xi_i, i \in \mathbb{Z})$ the probabilities $\mathbb{P}(A)$ and $\mathbb{P}(B)$ can be arbitrarily well approximated by $\mathbb{P}(A_n)$ and $\mathbb{P}(B_n)$ where

$$A^n, B^n \in \sigma(\xi_i, i \in \{-n, \ldots, n\})$$

for some $n \in \mathbb{N}$ large enough.

Indeed, let $\varepsilon > 0$ be given. By a standard exercise in measure theory there exists an $n \in \mathbb{N}$ and an event

$$A_n(\varepsilon) \in \sigma(\xi(i), i \in \{-n, \ldots, n\})$$

such that

$$\mathbb{P}(A \Delta A_n(\varepsilon)) < \varepsilon$$

(where for two sets A, A' we denote by $A \Delta A'$ the symmetric difference between A and A').

By the same arguments, for a given set $B \in \sigma(\xi(i))$ there exists $B_n(\varepsilon) \in \sigma(\xi(i), i \in \{-n, \ldots, n\})$ such that

$$\mathbb{P}(B \Delta B_n(\varepsilon)) < \varepsilon.$$

Hence by stationarity there also exists

$$B_n^s(\varepsilon) \in \sigma(\xi(i), i \in \{-n+s, \ldots, n+s\})$$

with

$$\mathbb{P}(\Theta^s(B) \Delta B_n^s(\varepsilon)) < \varepsilon.$$

(and indeed $B_n^s(\varepsilon)) = \Theta^s(B_n(\varepsilon)))$.

Again by stationarity (we may shift the whole situation by n), it suffices to assume that

$$A_n(\varepsilon), B_n(\varepsilon) \in \sigma(\xi(i), i \in \{0, \ldots, n\})$$

for some n large enough and hence that Hence we may without loss of generality assume that

$$B_n^s(\varepsilon) \in \sigma(\xi(i), i \in \{s, \ldots, n+s\}).$$

Then for $\varepsilon \leqslant \frac{1}{2}\mathbb{P}(A)$ we have

$$\mathbb{P}(A_n) = \mathbb{P}(A_n \cap A) + \mathbb{P}(A_n \setminus A) \leqslant \mathbb{P}(A) + \tfrac{1}{2}\mathbb{P}(A) = \tfrac{3}{2}\mathbb{P}(A)$$

and therefore

$$\begin{aligned} |\mathbb{P}(\Theta^t(B)|A) - \mathbb{P}(B)| &= \frac{|\mathbb{P}(\Theta^t(B) \cap A) - \mathbb{P}(B)\mathbb{P}(A)|}{\mathbb{P}(A)} \\ &\leqslant \frac{|\mathbb{P}(B_n^t(\varepsilon) \cap A_n(\varepsilon)) - \mathbb{P}(B_n(\varepsilon))\mathbb{P}(A_n(\varepsilon)| + 4\varepsilon}{\frac{2}{3}\mathbb{P}(A_n)} \\ &\leqslant \tfrac{3}{2}|\mathbb{P}(\Theta^t(B_n(\varepsilon))|A_n(\varepsilon)) - \mathbb{P}(B_n(\varepsilon))| + 6\varepsilon. \end{aligned}$$

Thus if we can show that $|\mathbb{P}(\Theta^t(B_n(\varepsilon))|A_n(\varepsilon)) - \mathbb{P}(B_n(\varepsilon))| \to 0$ we also know that $|\mathbb{P}(\Theta^t(B)|A) - \mathbb{P}(B)| \to 0$.

Thus it suffices to assume that

$$A, B \in \sigma(\xi_i, i \in \{0, \ldots, n\})$$

for some $n \in \mathbb{N}$ large enough.

Moreover, let us decompose $\mathbb{P}(B)$ in the following way:

$$\mathbb{P}(B) = \sum_{s \geqslant 0} \mathbb{P}(B| \text{ the last renewal before 0is at time} - s)$$
$$\times \mathbb{P}(\text{ the last renewal before 0 is at time} - s). \tag{5.1}$$

Similarly,

$$\mathbb{P}(\Theta^t(B)|A) = \sum_{s \geqslant 0} \mathbb{P}(\Theta^t(B)| \text{ the last renewal before } t \text{ is at time } t - s, A)$$
$$\times \mathbb{P}(\text{the last renewal before } t \text{ is at time } t - s \,|\, A). \tag{5.2}$$

Now

$$\begin{aligned}
&\mathbb{P}(\Theta^t(B)| \text{ the last renewal before } t \text{ is at time } t - s, A)\\
&\quad = \mathbb{P}(\Theta^t(B) \cap \text{ there is a renewal between times } n \text{ and } t|\\
&\qquad \text{the last renewal before } t \text{ is at time } t - s, A)\\
&\qquad + \mathbb{P}(\Theta^t(B) \cap \text{ there is no renewal between times } n \text{ and } t|\\
&\qquad \text{the last renewal before } t \text{ is at time } t - s, A)\\
&\quad = \mathbb{P}(B \cap \text{ there is a renewal between times } n - t \text{ and } 0|\\
&\qquad \text{the last renewal before 0 is at time} - s)\\
&\qquad + \mathbb{P}(\Theta^t(B) \cap \text{ there is no renewal between times } n \text{ and } t|\\
&\qquad \text{the last renewal before } t \text{ is at time } t - s, A)\\
&\quad = \mathbb{P}(B| \text{ the last renewal before 0 is at time} - s)\\
&\qquad - \mathbb{P}(B \cap \text{ there is no renewal between times } n - t \text{ and } 0|\\
&\qquad \text{the last renewal before 0 is at time} - s)\\
&\qquad + \mathbb{P}(\Theta^t(B) \cap \text{ there is no renewal between times } n \text{ and } t|\\
&\qquad \text{the last renewal before } t \text{ is at time } t - s, A),
\end{aligned}$$

where we have used the stationarity of $\mathbb{P}$.

Now we have a finite expected renewal time implying that as $t \to \infty$

$$\begin{aligned}
&\mathbb{P}(B \cap \text{ there is no renewal between times } n - t \text{ and } 0|\\
&\quad \text{the last renewal before 0 is at time} - s) \to 0
\end{aligned}$$

as well as

$$\mathbb{P}(\Theta^t(B) \cap \text{ there is no renewal between times } n \text{ and } t \,|$$
$$\text{the last renewal before } t \text{ is at time } t-s, A) \to 0.$$

This establishes equality between the first factors in (5.1) and (5.2).

For the second factors in (5.1) and (5.2) observe that due the same arguments by which we established the existence of a stationary measure

$$\mathbb{P}(\text{the last renewal before } t \text{ is at time } t-s \,|\, A)$$

converges independently of A (by which we mean that the limit is independent of A) to a number, which actually is

$$\mathbb{P}(\text{the last renewal before } 0 \text{ is at time } -s).$$

Hence the right-hand sides of (5.1) and (5.2) converge to each other as $t \to \infty$ yielding that

$$\mathbb{P}(\Theta^t(B) \,|\, A) \to \mathbb{P}(B)$$

as $t \to \infty$. Thus the distribution of the colors induced by π is stationary, mixing and hence ergodic.

Moreover, note that all v in V_3 have a positive probability of being in $\operatorname{Im} R$. However, still R is not transient. To understand why, observe that the scenery ξ in this example may be considered as a scenery drawn according to the distribution considered in Example 5.2, modified by random excursions of length $2l$. As already remarked in Example 5.2, R there is equivalent to a one-dimensional symmetric random walk. In terms of this random walk the excursions of length $2l$ may be interpreted as a holding. As the expected holding time is $\sum_{l \geqslant 0} 2l(1/2)^{l+1}$ and hence finite this holding does not spoil the recurrence of the random walk. Thus also in this example R is recurrent and therefore the condition of Theorem 2.1 is not fulfilled again.

At this point a little remark seems to be due.

Remark 5.4. Note that although in the examples above (in particular Examples 5.2 and 5.3) R is not transient and hence our main result Theorem 2.1 is not applicable, this does not mean that these sceneries cannot be reconstructed at all. As a matter of fact, the scenery in Example 5.2 has been proven to be reconstructible in Matzinger (1999b) by completely different methods and the same might hold true for the scenery in Example 5.3.

Before we give our main class of examples, let us quickly mention that there indeed are examples of two color sceneries that can be reconstructed with the help of Theorem 2.1.

Example 5.5. Like in Example 5.2 the easiest example of a two color scenery that can be reconstructed with the help of Theorem 2.1 is that of an i.i.d. biased scenery. More precisely, we choose

$$\mathbb{P}(\xi(0)=0)=p \quad \text{and} \quad \mathbb{P}(\xi(0)=1)=1-p,$$

(and, of course, $\mathbb{P}(\xi(0)=2)=0$) for some $p\in(0,1)$ with $p\neq\frac{1}{2}$, and let the $\xi(z), z\in\mathbb{Z}$ be i.i.d. Then R is equivalent to a one-dimensional random walk on the integers $\mathbb{Z}$ with drift. As, of course, such a random walk is transient, so is R and hence the main assumption of Theorem 2.1 is fulfilled and thus ξ can be reconstructed.

Let us now give our main class of examples. We will avoid the troubles we had in Example 5.3 by assuming that ξ is generated by a hidden Markov chain on a finite state space. To also avoid the troubles we had in Example 5.2 we additionally have to require that ξ is "essentially tree-like". Let us define this notion first.

Definition 5.6. We will call a class of sceneries *essentially tree-like*, if for their representation R the following holds:

$$\{v\in V_3\colon P(v\in \operatorname{Im} R)>0\}$$

consists of three distinct infinite branches. Thus, more precisely, we will call a class of sceneries essentially tree-like, if there is a vertex $v_0\in V_3$ such that

$$\{v\in V_3\colon P(v\in \operatorname{Im} R)>0\}\setminus\{v_0\}$$

has three infinite connected components.

Let us first show that Definition 5.6 is not empty.

Example 5.7. Let the random variables $\{\xi(z), z\in\mathbb{Z}\}$ be i.i.d. with

$$P(\xi(1)=0)>0, P(\xi(1)=1)>0 \quad \text{and} \quad P(\xi(1)=2)>0.$$

Then $\{v\in V_3\colon P(v\in \operatorname{Im} R)>0\}=V_3$ and thus the class of sceneries is essentially tree-like (one e.g. take the origin as v_0).

Remark 5.8. The notion essentially tree-like should not be confused with that a fixed scenery ξ has a representation $R=R(\xi)$ that has three distinct infinite branches. Indeed, the latter is never the case, because these branches would correspond to distinct, infinite, connected subset of $\mathbb{Z}$. Obviously, there are only two such subsets.

The counterexamples above show that the class of distribution we give below is the largest "natural class" for Theorem 2.1 to hold.

Theorem 5.9. *Consider the distribution of ξ produced by the following random mechanism. Take an aperiodic, irreducible, recurrent and stationary Markov chain $(X_n)_{n\in\mathbb{Z}}$ on a finite state space X. Let*

$$f: X\to\bigcup_{l\geqslant 1}\{(\zeta_1,\ldots,\zeta_l):\zeta_i\in\{0,1,2\}, i=1,\ldots,l\}$$

be a mapping from X the set of all words of finite length. Now select a scenery according to X_n and f. By this we mean, that we take a realization of X_n, and place

$f(X_0)$ to the integers $0,1,\dots,|f(X_0)|-1$, then we place $f(X_1)$ to the next integers and so on placing one word after the other. In the same way we color the negative integers by $(f(X_n)_{n\in\mathbb{Z}^-})$.

If then ξ is essentially tree-like, R is almost surely transient.

Example 5.10. This example (the simplest one can probably give) shows that the class of defined in Theorem 5.9 above is not empty: Every i.i.d. scenery, i.e. every scenery with i.i.d. colors, falls into the class described in Theorem 5.9. Indeed we simply take $X=\{0,1,2\}$ and $f(x)=x$ for $x=0,1,2$.. Moreover take the "independent" Markov chain on X, i.e. the Markov chain X_n with $\mathbb{P}(X_i=x)=\mathrm{P}_x\in(0,1)$ for all $x\in X$ and $n\in\mathbb{Z}$. Then it follows immediately that the corresponding scenery is essentially tree-like and obeys the conditions of Theorem 5.9.

Before we start to prove Theorem 5.9 let us define:

Definition 5.11. In the tree T_3 let us define the ball and the sphere of radius $r>0$ centered in some vertex $v\in V_3$, respectively, as

$$B(v,r):=\{w\in V_3\colon\ d(v,w)\leqslant r\}$$

and

$$S(v,r):=\{w\in V_3\colon\ d(v,w)=r\}.$$

Theorem 5.9 will be proved after the following lemma which justifies the notion "essentially tree-like" in the sense that the number of points that can possibly be visited by the scenery grows exponentially.

Lemma 5.12. *Under the conditions of Theorem 5.9 assume that $\operatorname{Im}R$ is essentially tree-like. Then there exists a constant $\kappa>0$ such that for each $v\in V_3$ and $r\in\mathbb{N}$ the ball of radius r centered in v, i.e. $B(v,r)\cap\{v\in V_3\colon P(v\in\operatorname{Im}R)>0\}$, contains at least $\mathrm{e}^{\kappa r}$ vertices.*

Proof. To show this lemma we will prove that under these conditions $\operatorname{Im}R$ is indeed an infinitely branching tree by exploiting the essential self-similarity of $\operatorname{Im}R$. By the latter we mean that given two vertices $v_1,v_2\in\operatorname{Im}R$ such that both are, for example, colored by reading the endpoint of a word $f(x),x\in X$, then the neighborhoods of v_1 and v_2 are isomorphic.

So, if $\operatorname{Im}R$ is essentially tree-like, by definition, it will contain three disjoint, infinite branches b_1,b_2,b_3 and without loss of generality we can assume that $v_0=o$, i.e. the split point is assumed to be the origin. For convenience let $X=\{x_0,\dots,x_l\}$, $f(x_0)=(\zeta^1,\dots,\zeta^\lambda)$ where $\zeta^i\in\{0,1,2\}$, $(i=1,\dots,\lambda)$, and assume that the color of o is produced by reading ζ^1. This means we read the color of the origin in the first letter of the word belonging to x_0.

Define $L=\sum_{i=0}^{l}|f(x_i)|$.

Now $(X_n)_{n\in\mathbb{N}}$ is a stationary and recurrent Markov chain on X. This immediately implies that

$$\mathbb{P}(X_n = x_j \mid X_1 = x_i) > 0 \quad \text{for some } n \leqslant |X| = l+1$$

and all $x_i, x_j \in X$ (because have to be able to come back to x_i from some point in X). In particular,

$$\mathbb{P}(X_n = x_i \mid X_1 = x_i) > 0 \quad \text{for some } n \leqslant |X| = l+1$$

and all $x_i \in X$.

Recall that each x_i in X produces a word $f(x_i)$. Say, we find this word in the scenery, starting in $z_0 \in \mathbb{Z}$. The above considerations imply that we have a positive probability to see $f(x_i)$ again the latest every L steps. This implies that for all $v_1 \in \operatorname{Im} R$ there is a $v_2 \neq v_1 \in \operatorname{Im} R$ such that $d(v_1, v_2) \leqslant L$ and the color of v_2 as read at the same position of the same word as the color of v_1. Otherwise the random path R would return to v_1 every L steps contradicting the assumption that it is infinite. Thus for every point $v \in V_3$ every possible situation, i.e. every color read at any position of any of the words, occurs within a ball of radius L.

We will apply these considerations to the origin o. We take two auxiliary points $a_1, a_2, a_3 \in V_3$ with $a_i \in b_i$ (recall that b_i was the i'th branch) and $d(o, a_i) = 2L$ for $i = 1, 2, 3$. Applying the above shows that there are vertices $v_1, v_2, v_3 \in V_3$ with $v_i \in b_i$ and $d(v_i, a_i) \leqslant L$ for $i = 1, 2, 3$, such that the color of v_i is read by R at ζ^1. Obviously, $v_i \neq o$ for each i. On the other hand, the situation at v_i is the same as at o, that means, in particular, at v_i there are three different infinite branches for all i.

Continuing inductively yields the desired result. $\square$

Now we are ready to prove that R is transient.

Proof of Theorem 5.9. The basic idea of the proof will be to show that, if R were recurrent, then for any fixed vertex $v \in \operatorname{Im} R$ the distance $d(v, R(n))$ would be stochastically bounded below by a random walk with positive drift. This, of course, is a contradiction, since a random walk with positive drift is transient and thus R would also have to be transient.

The way to derive this contradiction is to first analyze a Markov chain that is obtained from R by stopping it, when is has moved to the next point a certain distance apart from the previous point. We will see that this Markov chain has the tendency to move away from the points it has previously visited. By comparing this chain to a (transient) random walk with drift on the line we see that it is transient as well. But R and this chain are never far apart from each other. Hence also R is transient.

More precisely in what follows take $d_1 < d_2 \in \mathbb{N}$ and typically we will think of d_1 as being much smaller than d_2 (a more precise description will be given below).

Again let $X = \{x_0, \ldots, x_l\}$ and $f(x_0) = (\zeta^1, \ldots, \zeta^\lambda)$ where $\zeta^i \in \{0, 1, 2\}$, $(i = 1, \ldots, \lambda)$, and take $L = \sum_{i=0}^{l} |f(x_i)|$. Consider the following Markov chains induced by R, X_n, and ξ. Take

$$\Omega = V_3 \times X \times \left\{1, \ldots, \max_{i \in \{0,\ldots,l\}} |f(x_i)|\right\}$$

and

$$\tilde{X}_n := (X_{n,1}, X_{n,2}, X_{n,3}) := (R(n), w, k),$$

where $w \in X$ is the word where $R(n)$ is read, and k is the position in w where $R(n)$ is read. Note that $\tilde{X}_n$ is again a Markov chain. Moreover, introduce a sequence of stopping times $(t_n)_{n\in\mathbb{Z}}$ such that $t_0 = 0$ and

$$t_n := \inf\{t > t_{n-1}, d(R(t), R(t_{n-1})) = d_2\}, \quad n \in \mathbb{N}$$

and

$$t_{-n} := \sup\{t < t_{-n+1}, d(R(t), R(t_{-n+1})) = d_2\}, \quad n \in \mathbb{N}.$$

Let $\tilde{Y}_t = (Y_{t,1}, Y_{t,2}, Y_{t,3})$ where for $t \in [t_n, t_{n+1})$ $Y_{t,1} = X_{t_n,1}$, $Y_{t,2} = X_{t,2}$ and $Y_{t,3} = X_{t,3}$. Note that $\tilde{Y}_t$ inherits the Markov property from $\tilde{X}_t$. We will prove that $\tilde{Y}_t$ is transient. This also implies that $R(t)$ is transient since

$$d(R(t), Y_{t,1}) \leqslant d_2$$

and d_2 is independent of t.

Let for any $n \in \mathbb{Z}$ denote $y_n := Y_{t_n,1}$. The transience of y_n will be proved by showing that, if y_n were recurrent, for any fixed vertex $v_0 \in V_3$ the increment $d(v_0, y_{n+1}) - d(v_0, y_n)$ would be stochastically bounded below by the increment of a random walk with positive drift. That random walk can go by a distance d_2 to the left with probability $\frac{1}{4}$ and go by $d_2/2$ to the right with probability $\frac{3}{4}$. Let $n_0 \in \mathbb{N}$ be fixed and without loss of generality in the following we will assume that the color of y_{n_0} (that is $\varphi(y_{n_0})$) is read at ζ_l (that is, it is equal to ζ_l and read at the last position of $f(x_0)$).

Let us assume that $4 \mid d_2$ and take the unique point z_0 such that $d(y_{n_0}, z_0) = \frac{1}{4}d_2$ and $d(v_0, z_0) = d(v_0, y_{n_0}) - \frac{1}{4}d_2$ (see Fig. 2 below). Denote

$$\Xi := \{v \in S(y_{n_0}, d_2) \setminus S(z_0, \tfrac{3}{4}d_2)\}.$$

We have that for such $v \in \Xi$

$$d(v, v_0) \geqslant d(y_{n_0}, v_0) + d_2/2.$$

To understand this and the following, one should keep in mind that Fig. 2 illustrates the tree geometry in T_3. Thus a path from v_0 to $v \in \Xi$ has to follow the path from v_0 to y_{n_0} at least until z_0. On the other hand, for

$$v \in S(y_{n_0}, d_2) \setminus \Xi$$

we have by the triangle inequality

$$d(v, v_0) \geqslant d(y_{n_0}, v_0) - d_2.$$

Hence, if we can, for example, show that

$$\mathbb{P}(y_{n_0+1} \in S(y_{n_0}, d_2) \cap S(z_0, \tfrac{3}{4}d_2)) \leqslant \tfrac{1}{4},$$

we are done, since then $d(y_{n_0+1}, v_0)$ can be smaller than $d(y_{n_0}, v_0)$ by d_2 with probability at most $\frac{1}{4}$. On the other hand, it will increase by $d_2/2$. Thus, the increment $d(y_{n_0+1}, v_0) - d(y_{n_0}, v_0)$ is bounded below by the increment of a random walk, which

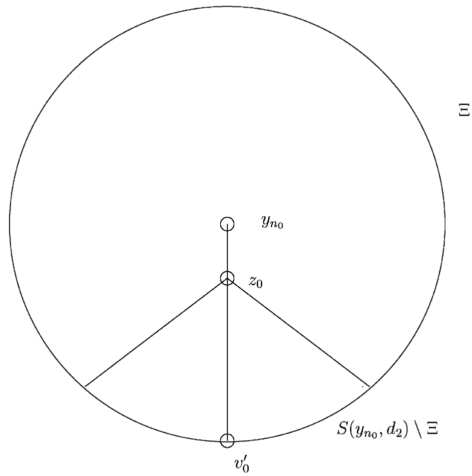

Fig. 2. Set Ξ.

at each step can do the following: go to the left by a distance d_2 with probability $\frac{1}{4}$ or go to the right by a distance $d_2/2$ with probability $\frac{3}{4}$. We thus get

$$\mathbb{E}(d(y_{n_0+1}, v_0) - d(y_{n_0}, v_0)) \geqslant \left(-\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{4}\right) d_2 = \frac{1}{8}\, d_2.$$

In order to bound $\mathbb{P}(y_{n_0+1} \in S(y_{n_0}, d_2) \cap S(z_0, \frac{3}{4}d_2))$ we will use Lemma 5.12 above. The idea is that Lemma 5.12 tells us that in a neighborhood of y_{n_0} there are many points with the same color as y_{n_0} and, where this color is read at the same position of the same position as y_{n_0}. Hence the situation in any of these points is the same as in y_{n_0}. On the other hand (as we will prove), most of these points belong to disjoint "first exit regimes" of $S(y_{n_0}, d_2)$. Since the situation in all of the points is the same, the probabilities to leave the circle $S(y_{n_0}, d_2)$ via a particular of the corresponding segments are about the same. In particular, the probability to leave $S(y_{n_0}, d_2)$ via $S(y_{n_0}, d_2) \setminus \Xi$ cannot be too large.

More precisely, in order to bound $\mathbb{P}(y_{n_0+1} \in S(y_{n_0}, d_2) \cap S(z_0, \frac{3}{4}d_2))$ we consider the ball $B(y_{n_0}, d_1)$ with $d_1 \ll d_2$. As we have shown in Lemma 5.12 this ball contains exponentially many points (in d_1). By the same argument one can show that there exists a sphere inside $B(y_{n_0}, d_1)$ containing $M := e^{\kappa d_1}$ many points (for some $\kappa > 0$) the color of which is read at the same point ζ^λ. Let us call these points $v_1, \dots, v_M$.

Now let us assume that our proposition was wrong and R was recurrent. Then for any given $\varepsilon > 0$ we could choose d_2 large enough such that with probability larger than $1 - \varepsilon$ the random path R will visit each point inside a ball of radius d_1 around its starting point before first exiting a ball of radius $d_2 - d_1$ around the starting point.

Instead of bounding now

$$\mathbb{P}(y_{n_0+1} \in S(y_{n_0}, d_2) \cap S(z_0, \tfrac{3}{4}d_2))$$

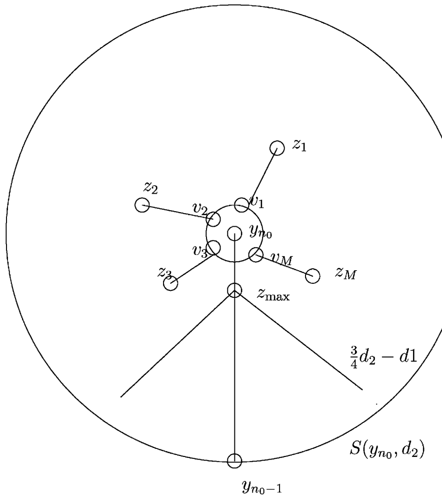

Fig. 3. Points $v_1, \ldots v_M$ and $y_1, \ldots y_M$.

we will rather bound

$$\Delta := \max_{z \in S(y_{n_0}, \frac{1}{4}d_2)} \mathbb{P}(y_{n_0+1} \in S(y_{n_0}, d_2) \cap S(z, \tfrac{3}{4}d_2)).$$

This means, we will bound the maximum of the probabilities to first exit $S(y_{n_0}, d_2)$ via a segment similar to $S(y_{n_0}, d_2) \cap S(z, \frac{3}{4}d_2)$. In order to do so, we will bound

$$\Delta' := \max_{z \in S(y_{n_0}, \frac{1}{4}d_2)} \mathbb{P}(y'_{n_0+1} \in S(y_{n_0}, d_2 - d_1) \cap S(z, \tfrac{3}{4}d_2 - d_1)),$$

where y'_{n_0+1} is the vertex where the path R first exits $S(y_{n_0}, d_2 - d_1)$ when starting in y_{n_0}. Note that both maxima are actually attained and let $z_{\max}$ be such that

$$\Delta' := \mathbb{P}(y'_{n_0+1} \in S(y_{n_0}, d_2 - d_1) \cap S(z_{\max}, \tfrac{3}{4}d_2 - d_1)).$$

Note that the stretch $\overline{y_{n_0} z_{\max}}$ is uniquely defined by its corresponding sequence of colors and its starting point y_{n_0}. Since the situation is completely identical in each of the points $v_1, \ldots, v_M$ and in y_{n_0} we can find $z_1, \ldots, z_M$ that correspond to $z_{\max}$ when we replace y_{n_0} by $v_1, \ldots, v_M$. More precisely the points $z_1, \ldots, z_M$ are defined by the fact that the stretch $\overline{v_i z_i}$ has the same color sequence as $\overline{y_{n_0} z_{\max}}$, (for each $i = 1, \ldots, M$). (Fig. 3).

Obviously, the probability of first exiting $S(v_i, d_2 - d_1)$ via $S(v_i, d_2 - d_1) \cap S(z_i, \frac{3}{4}d_2 - d_1)$ when starting in v_i also equals Δ'. Let us call z_i and z_j equivalent, if the stretches $\overline{v_i z_i}$ and $\overline{v_j z_j}$ intersect inside $B(y_{n_0}, d_1)$. Note that the stretches $\overline{v_i z_i}$ and $\overline{v_j z_j}$ always decrease there distance to $S(y_{n_0}, d_2)$. Hence, if z_i and z_j are not equivalent, then the stretches $\overline{v_i z_i}$ and $\overline{v_j z_j}$ do not intersect at all and z_i and z_j and $S(v_i, d_2 - d_1) \cap S(z_i, \frac{3}{4}d_2 - d_1)$ and $S(v_j, d_2 - d_1) \cap S(z_j, \frac{3}{4}d_2 - d_1)$ have distance at least $2d_2 - 2d_1$. Moreover, observe that for a given $i \in \{1, \ldots, M\}$ at most $2d_1$ of the $\{z_1, \ldots, z_M\}$ can be equivalent to z_i. This follows from the fact that two stretches $\overline{v_i z_i}$ and $\overline{v_j z_j}$ can only intersect at different "times" (that is their distance from v_i—or v_j, respectively, —must be different). As each stretch has to leave $B(y_{n_0}, d_1)$ after at most $2d_1$ steps, there, indeed can be at most $2d_1$ of the z_j equivalent to z_i. As M is an exponential in d_1, but $2d_1$ is of course only linear, we can have as many non-equivalent z_i's as we wish. We will denote the number of non-equivalent z_i's by $M' \in \mathbb{N}$.

Now, once we are first exiting $S(y_{n_0}, d_2 - d_1)$ via $S(y_{n_0}, d_2 - d_1) \cap S(z_{\max}, \frac{3}{4}d_2 - d_1)$ we are at distance at d_1 from $S(y_{n_0}, d_2)$. Define

$$\mathscr{Z} := \{z \in V_3\colon\ d(y_{n_0}, z) = \tfrac{1}{4}d_2, d(z_{\max}, z) = d_1\}.$$

By construction of d_1 the probability of visiting each point in a ball of radius d_1 before leaving a ball of radius $d_2 - d_1$ for the first time is at least $1 - \varepsilon$. Hence, once we are in $S(y_{n_0}, d_2 - d_1) \cap S(z_{\max}, \frac{3}{4}d_2 - d_1)$ the probability to leave $S(y_{n_0}, d_2 - d_1)$ for the first time via

$$\bigcup_{z \in \mathscr{Z}} S(y_{n_0}, d_2) \cap S(z, \tfrac{3}{4}d_2)$$

is at least $1 - \varepsilon$. Since ε can be arbitrarily small this shows in particular that

$$\Delta(1 - \varepsilon) \leqslant \Delta'.$$

Moreover, introduce a random variable N, which is equal to $i \in \{1, \dots M'\}$ if $R(t)$ visits $S(v_i, d_2 - d_1) \cap S(z_i, \frac{3}{4}d_2 - d_1)$ before visiting any of the $S(v_j, d_2 - d_1) \cap S(z_j, \frac{3}{4}d_2 - d_1), j \neq i$ and exiting $S(y_{n_0}, d_2)$ for the first time. If $R(t)$ does not visit any of the $S(v_i, d_2 - d_1) \cap S(z_i, \frac{3}{4}d_2 - d_1)$ before exiting $S(y_{n_0}, d_2)$ for the first time we set N equal to 0. Now, once $R(t)$ is in v_i the probability that $P(N_i = i)$ is at least Δ' (by construction of the v_i and z_i). On the other hand, $R(t)$ hits with probability at least $1 - \varepsilon$ any of the v_i before exiting $S(y_{n_0}, d_2 - d_1)$ for the first time, i.e. before the value of N can be determined. This implies

$$1 \geqslant \sum_{i=1}^{M'} P(N = i) \geqslant M'(1 - \varepsilon)\Delta'.$$

Since M' can be made as large as we wish, Δ' and hence also Δ are as small as we wish, for example less than $\frac{1}{4}$. Therefore, under the assumption that R is recurrent,

$$\mathbb{E}(d(y_{n_0+1}, y_{n_0-1})) \geqslant \tfrac{9}{8}d_2.$$

But this implies that $Y_{n,1}$ is transient. Indeed, by the tree structure of T_3 the distance of $Y_{n,1}$ to any fixed point v' is stochastically larger than the distance to the origin of a homogeneous random walk Z_n on the integer line $\mathbb{Z}$ with jump length d_2 and

$$\mathbb{P}(Z_{n+1} = z + d_2 | Z_n = z) = 1 - \mathbb{P}(Z_{n+1} = z - d_2 | Z_n = z) = \tfrac{3}{4}$$

for all $n \in \mathbb{N}$ and $z \in \mathbb{Z}$. This is a contradiction.

Hence $Y_{n,1}$ is transient and therefore also R is transient, which is what we claimed. □

The next theorem basically states that under the conditions of Theorem 5.9 the random path R (the representation of ξ as a random path on T_3) is not only transient but also has positive speed with probability exponentially close to one. Theorem 5.13 will be the basis for an exceptionally good test for distinguishing two sceneries.

Theorem 5.13. *Under the conditions of Theorem* 5.9 (*in particular we also assume that* ξ *is essentially tree-like*) *there exist constants* $c_0, c_1, c_2 > 0$ *such that for all* $n \in \mathbb{N}$ *and every fixed* $v_0 \in V_3$

$$\mathbb{P}(\min\{d(v_0, R(n)), d(R(-n), v_0)\} \leqslant c_2 n) \leqslant c_0 \mathrm{e}^{-c_1 n}.$$

Proof. The proof is intrinsically related to the proof of Theorem 5.9. We will make use of the notations introduced there. Let $Y'_m := X_{t_m,1}$. Let us (for a moment) assume that for any fixed vertex v_0 the distance $(d(v_0, Y'_m))_m$ can be stochastically bounded below by the distance to the origin of a random walk on the line with drift. Hence the existence of constants $c'_0, c'_1, c'_2 > 0$ such that

$$\mathbb{P}(\min\{d(v_0, Y'_m) \leqslant c'_2 m) \leqslant c'_0 \mathrm{e}^{-c'_1 m} \tag{5.3}$$

follows immediately from an exponential estimate for this dominating random walk.

In order to conclude the desired result from (5.3), we need to understand that there exist constants $c_3, c_4 > 0$ such that

$$\mathbb{P}(t_n \geqslant c_3 n) \leqslant \mathrm{e}^{-c_4 n} \tag{5.4}$$

(recall that t_n was the nth stopping time). But this follows form decomposing t_n into

$$t_n = (t_n - t_{n-1}) + (t_{n-1} - t_{n-2}) + \cdots + (t_1 - t_0).$$

By the Markov property of $\tilde{X}_t$ and $\tilde{Y}_t$ the random variables $(t_m - t_{m-1})$ are stochastically independent. Also, since the state space X of the Markov chain X_n is finite all these $(t_m - t_{m-1})$ can be stochastically dominated from above by a random variable $T_{\max}$ (describing the exit time of $\tilde{X}_t$ from a circle of radius d_2 when starting in the state with the longest exit time). Now $T_{\max}$ has a finite moment generating function

$$\mathbb{E}\mathrm{e}^{\theta T_{\max}} < \infty. \tag{5.5}$$

This follows, since X is finite. Therefore, we can find constants $L_1 < \infty$ and $p > 0$ such that independent of where we start with $R(n)$ we have hit the sphere of radius d_2 centered in the starting point after L_1 steps with probability at least $p > 0$. This implies (5.5) and therefore by large deviation estimates also (5.4).

It remains to show that for any fixed vertex v_0 the distance $(d(v_0, Y'_m))_m$ can be stochastically bounded below by the distance to the origin of a random walk on the line with drift.

Lemma 5.14 below states that the maximum probability for R to ever reach a point at distance d from v_0 when starting in v_0 tends to zero as d goes to infinity. More precisely, it says that for every (t_0, v_0, w_0, k_0) with

$$\mathbb{P}(\tilde{X}_{t_0} = (v_0, w_0, k_0)) > 0$$

we have that

$$\lim_{d \to \infty} \max_{v: d(v, v_0) = d} \mathbb{P}(\exists s > 0\colon R(t_0 + s) = v | \tilde{X}_{t_0} = (v_0, w_0, k_0)) = 0. \tag{5.6}$$

Assuming that Lemma 5.14 below is true, we choose d_2 large enough such that for $d = \frac{1}{4}d_2$ we have

$$\max_{v:d(v,v_0)=d} \mathbb{P}(\exists s > 0\colon\ R(t_0+s) = v | \tilde{X}_{t_0} = (v_0, w_0, k_0)) \leqslant \tfrac{1}{10}.$$

Let Z_n be a random walk on the line that in each step either steps to the left by d_0 (this happens with probability $\frac{1}{10}$) or steps to the right by $\frac{1}{2}d_0$ (this happens with probability $\frac{9}{10}$). Then $(d(v_0, Y_m'))_m$ can be stochastically bounded below by the distance of Z_n to the origin. Indeed, for fixed n_0 let z_0 denote the unique vertex in V_3 at distance d from y_{n_0} and at distance $3d$ from y_{n_0-1}. If, after time t_{n_0} the random path R never visits z_0 the point y_{n_0+1} is at distance at least $2d = \frac{d_2}{2}$ from y_{n_0}. This happens with probability $\frac{9}{10}$. By the tree structure of T_3 this shows that indeed $(d(v_0, Y_m'))_m$ can be stochastically bounded below by the distance to the origin of a random walk on the line with drift.

Thus the statement of the theorem follows. □

It remains to show

Lemma 5.14. *For all* (t_0, v_0, w_0, k_0) *with*

$$\mathbb{P}(\tilde{X}_{t_0} = (v_0, w_0, k_0)) > 0$$

we have that

$$\lim_{d\to\infty} \max_{v:d(v,v_0)=d} \mathbb{P}(\exists s > 0\colon\ R(t_0+s) = v | \tilde{X}_{t_0} = (v_0, w_0, k_0)) = 0.$$

Proof. Assume Lemma 5.14 was wrong. Then there exists a sequence $v_0, v_1, v_2 \ldots \in V_3$ such that v_i, v_{i+1} are neighbors (for all $i \in \mathbb{N}_0$) but $v_i \neq v_{i+2}$ (for all $i \in \mathbb{N}_0$) such that

$$\lim_{n\to\infty} \mathbb{P}(\exists s > 0\colon\ R(t_0+s) = v_n | \tilde{X}_{t_0} = (v_0, w_0, k_0)) > 0.$$

Hence for

$$A := \bigcap_{n\in\mathbb{N}} \{\exists s > 0\colon\ R(t_0+s) = v_n\}$$

it holds

$$\mathbb{P}(A | \tilde{X}_{t_0} = (v_0, w_0, k_0)) > 0.$$

But this implies that

$$\mathbb{P}(A | \tilde{X}_{t_0} = (v_0, w_0, k_0)) = 1.$$

Indeed, otherwise $\mathbb{P}(A | \tilde{X}_{t_0} = (v_0, w_0, k_0)) \in (0,1)$ and hence already

$$\mathbb{P}\left(\bigcap_{n\leqslant N} \{\exists s > 0\colon\ R(t_0+s) = v_n\} | \tilde{X}_{t_0} = (v_0, w_0, k_0)\right) =: p \in (0,1)$$

for some $N \in \mathbb{N}$. But then we could stop R the first time that $(\tilde{X}_{t,2}, \tilde{X}_{t,3}) = (w_0, k_0)$ (which happens infinitely often since the original Markov chain (X_n) on X is recurrent), after $R(t)$ has visited v_N. This first segment of points has probability at most p. But at $(\tilde{X}_{t,2}, \tilde{X}_{t,3}) = (w_0, k_0)$ the situation is the same as in (v_0, w_0, k_0), so again we find a finite segment of the sequence of v_i's with probability at most p and so on. This shows that if $\mathbb{P}(A|\tilde{X}_{t_0} = (v_0, w_0, k_0)) \in (0,1)$ we already know that $\mathbb{P}(A|\tilde{X}_{t_0} = (v_0, w_0, k_0)) = 0$.

Moreover, there only can be one sequence $v_0, v_1, v_2 \ldots \in V_3$ such that v_i, v_{i+1} are neighbors (for all $i \in \mathbb{N}_0$) but $v_i \neq v_{i+2}$ (for all $i \in \mathbb{N}_0$) and $\mathbb{P}(A|\tilde{X}_{t_0} = (v_0, w_0, k_0)) = 1$. This follows from the tree structure of T_3. Indeed, if there were two such sequences they eventually needed to be on disjoint branches of T_3. But then R in order to visit both sequences with probability one needs to visit the bifurcation point of the two sequences infinitely often (contradicting its transience).

Eventually we show that also $\mathbb{P}(A|\tilde{X}_{t_0} = (v_0, w_0, k_0)) = 1$ cannot hold. Again we exploit that R is essentially tree-like.

Let us call a sequence essentially k-periodic, if it is k-periodic up to a finite number of elements. By definition the quasi-period k of an essentially k-periodic sequence is the period of the periodic part that sequence. Now recall that the colors $\varphi(v_0), \varphi(v_1), \varphi(v_2), \ldots$ are produced by a Markov chain on *finite* state space X. Since moreover each word $f(x), x \in X$ had a finite length, there is just a finite number of possible positions for the second and third coordinate of the process $\tilde{X}_t$. Since the situation is the same, whenever the second and the third coordinate of the process $\tilde{X}_t$ are in the same point and there is just one sequence $v_0, v_1, v_2 \ldots \in V_3$ satisfying the above conditions, the sequence $\varphi(v_0), \varphi(v_1), \varphi(v_2), \ldots$ is essentially periodic. Since we do not care for a finite number of elements we can and will assume that it is periodic.

Now R is essentially tree-like. This means in particular that we can find a point $v'_0 \in V_3$ with the following properties.

- $v'_0 \notin \{v_0, v_1, v_2, \ldots\}$.
- $\mathbb{P}(\exists s > 0\colon R(t_0 + s) = v'_0 | \tilde{X}_{t_0} = (v_0, w_0, k_0)) > 0$.
- $\varphi(v_0) = \varphi(v'_0)$ and moreover the color of v_0 and v'_0 are read in the same word at the same position.
- $d(v'_0, \{v_0, v_1, v_2, \ldots\}) \geqslant 3L$, where $L := \sum_{x \in X} |f(x)|$. This latter condition ensures that for all but possibly the first L colors the sequence $\varphi(v'_0), \varphi(v'_1), \varphi(v'_2), \ldots$ follows the same period as $\varphi(v_0), \varphi(v_1), \varphi(v_2), \ldots$.

Let us take such a v'_0. Since the situation in v'_0 is completely identical to the situation in v_0, there exists a sequence $v'_0, v'_1, v'_2, \ldots$ such that v'_i, v'_{i+1} are neighbors (for all $i \in \mathbb{N}_0$) but $v'_i \neq v'_{i+2}$ (for all $i \in \mathbb{N}_0$) such that

$$\lim_{n \to \infty} \mathbb{P}(\exists s > 0\colon R(t_0 + s) = v'_n | \tilde{X}_{t_0} = (v'_0, w_0, k_0)) = 1.$$

But then the sequences $v_0, v_1, v_2 \ldots$ and $v'_0, v'_1, v'_2, \ldots$ have to merge. Otherwise (since v'_0 van be reached from v_0 with positive probability) from v_0 there would be two disjoint sequences that are both visited with probability one in contradiction to what was shown above. Say the sequences $v_0, v_1, v_2 \ldots$ and $v'_0, v'_1, v'_2, \ldots$ merge in $v_n \in \{v_0, v_1, v_2 \ldots\}$. Then for some n' and all $j \in \mathbb{N}$

$$v_{n+j} = v'_{n'+j}$$

and also

$$(v_{n+j}) = \varphi(v'_{n'+j}).$$

However,

$$\varphi(v_{n-1}) \neq \varphi(v'_{n'-1}),$$

since v_n can only have one neighbor of the color $\varphi(v_{n-1})$. Hence $n \neq n'$. Hence by the above the sequence $\varphi(v'_0), \varphi(v'_1), \varphi(v'_2), \ldots$ has quasi-period $n - n'$. On the other hand, since we have chosen $d(v'_0, \{v_0, v_1, v_2, \ldots\}) \geqslant 3L$, the quasi-periodic sequence $\varphi(v'_0), \varphi(v'_1), \varphi(v'_2), \ldots$ in $n' - 1$ is already in its "periodic part". But then its period is the same as that of $\varphi(v_0), \varphi(v_1), \varphi(v_2), \ldots$, in particular

$$\varphi(v_{n-1}) = \varphi(v'_{n'-1}),$$

This is a contradiction. Hence the lemma is true. □

Theorem 5.13 is extremely useful when we try to attack one of the original problem of this area, that is the scenery distinguishing problem, where we have to tell from the color record on which of two sceneries the this color record has been produced. Of course, in principal, already Theorem 5.9 shows that we can distinguish two sceneries drawn independently and at random from a distribution satisfying the conditions of Theorem 5.9. On the other hand, for all possible applications this test is not practicable. We now show that, indeed, as a consequence of Theorem 5.13 there is a practicable test based on very little information that works "exponentially well".

More precisely, we show that given any scenery ξ randomly drawn from a distribution satisfying the conditions of Theorems 5.9 and 5.13 and another scenery η of which we know nothing at all, there is a test that works exponentially well for a set of sceneries with probability exponentially close to one. Here the notions "exponentially well" and "with probability exponentially close to one" stand for the following. Given that we know $\chi|[0,n]$ (so the first $n+1$ observations) and, for example, two points in the representation of ξ, namely

$$v = R(m^+) \quad \text{where } m^+ := \min\{m \in \mathbb{N}\colon d(o, R(m)) \geqslant n^{1/3}\}$$

and

$$w = R(m^-) \quad \text{where } m^- := \max\{m \in \mathbb{Z}_-\colon d(o, R(m)) \geqslant n^{1/3}\}$$

(where, as above, $R(\cdot) = R(\xi, \cdot)$ is the representation of ξ on T_3) we can find a test, which for a subset of ξ's of probability larger than $1 - k_0 \mathrm{e}^{-k_1 n^{1/3}}$ has failure probability less than $\mathrm{e}^{-k_2 n^{1/3}}$. Let us formalize this in a theorem:

Theorem 5.15. *Let ξ be randomly drawn from a distribution satisfying the conditions of Theorems* 5.9 *and* 5.13 *and let η be another scenery of which we know nothing at all. Assume that we know $\chi|[0,n]$,*

$$v = R(m^+) \quad \textit{where } m^+ := \min\{m \in \mathbb{N}\colon d(o, R(m)) \geqslant n^{1/3}\}$$

and

$$w = R(m^-) \quad \textit{where } m^- := \max\{m \in \mathbb{Z}_-\colon d(o, R(m)) \geqslant n^{1/3}\},$$

(*where, as above*, $R(\cdot) = R(\xi, \cdot)$ *is the representation of* ξ *on* T_3). *Then we can find a test*

$$T : \{0,1,2\}^{n+1} \times V_3 \times V_3 \to \{\xi, \eta\}$$

constants $k_0, k_1, k_2, k_3 > 0$ *and a set* $\bar{\Xi} \subset \{0,1,2\}^{\mathbb{Z}}$ *with*

$$\mathbb{P}(\bar{\Xi}) \geqslant 1 - k_0 \mathrm{e}^{-k_1 n^{1/3}}$$

such that, whenever $\xi \in \bar{\Xi}$

$$\mathbb{P}(T(\chi|[0,n], v, w) = \xi | \, \chi \text{ has been produced on } \eta) = 0$$

and

$$\mathbb{P}(T(\chi|[0,n], v, w) = \eta | \chi \text{ has been produced on } \xi) \leqslant k_2 \mathrm{e}^{-k_3 n^{1/6}}.$$

Remark 5.16. Note that we are able to improve the known tests for scenery distinguishing in the following important features:

(1) ξ can be drawn from a large class of distributions admitting correlations between the color of different sites.
(2) η can be arbitrary.
(3) No knowledge is required about η.
(4) Only very limited knowledge about ξ is required

Proof of Theorem 5.15. For fixed η we propose is the following test T:

Whenever $R \circ S$ reaches $v = v(\xi)$ or $w(\xi)$ within $[0,n]$ we say that the scenery is ξ, otherwise we say that it is η.

$\bar{\Xi}$ will be the set of sceneries ξ such that we have a fair chance to see v and w in the first n observations and that $v(\xi)$ and $w(\xi)$ are different from any $v(\eta)$ and $w(\eta)$. Formally we define the following set of sceneries. For a scenery ξ let

$$m^+ := m^+(\xi) := \min\{m \in \mathbb{N}\colon\ d(o, R(m)) \geqslant n^{1/3}\}$$

and

$$m^- := m^+(\xi) := \max\{m \in \mathbb{Z}_-\colon\ d(o, R(m)) \geqslant n^{1/3}\}$$

then for some constant $\kappa > 0$ (note that R depends on ξ)

$$\bar{\Xi} := \{\xi\colon\ \max m^+, |m^-| \leqslant \kappa n^{1/3}, R(i) \notin \{v(\eta), w(\eta)\} \ \text{ for } i = -n, -n+1, \dots, n\}. \tag{5.7}$$

Here $R(i)$ is the images of the scenery ξ in the lattice point $i \in \mathbb{Z}$. Then for κ large enough and some constants $k_0, k_1 > 0$ it holds

$$\mathbb{P}(\bar{\Xi}) \geqslant 1 - k_0 \mathrm{e}^{-k_1 n^{1/3}}. \tag{5.8}$$

Indeed, applying Theorem 5.13 to the origin $v_0 = o$ shows that

$$\mathbb{P}(\{\xi\colon\ \max m^+, |m^-| \geqslant \kappa n^{1/3}\}) \leqslant k'_0 \mathrm{e}^{-k'_1 n^{1/3}} \tag{5.9}$$

for κ large enough and some constants $k'_0, k'_1 > 0$. On the other hand, trivially

$$\mathbb{P}(R(i) \notin \{v(\eta), w(\eta)\}) = 1$$

for $i = -n^{1/3} + 1, \ldots, n^{1/3} - 1$ and any ξ. Moreover, applying Theorem 5.13 to the $v_0 = v(\eta)$ gives

$$\mathbb{P}(R(i) = v(\eta)) \leqslant k_0'' \mathrm{e}^{-k_1'' n^{1/3}}$$

for $|i| \geqslant n^{1/3}$ and similarly

$$\mathbb{P}(R(i) = w(\eta)) \leqslant k_0'' \mathrm{e}^{-k_1'' n^{1/3}}$$

for $|i| \geqslant n^{1/3}$ for some constants $k_0'', k_1'' > 0$. So altogether

$$\begin{aligned}\mathbb{P}(\xi\colon \exists i \in \{-n, -n+1, \ldots, n\}\colon R(i) \in \{v(\eta), w(\eta)\}) &\leqslant 2nk_0'' \mathrm{e}^{-k_1'' n^{1/3}} \\ &\leqslant k_0''' \mathrm{e}^{-k_1''' n^{1/3}}\end{aligned}$$

for some constants $k_0''', k_1''' > 0$. Together with (5.9) this implies (5.8).

If now $\xi \in \bar{\Xi}$, then indeed

$$\mathbb{P}(T(\chi|[0,n], v, w) = \xi|\, \chi \text{ has been produced on } \eta) = 0,$$

since $T(\chi|[0,n], v, w) = \xi$ if and only if we read $v(\xi)$ or $w(\xi)$ and by construction this cannot happen on η. On the other hand, $T(\chi|[0,n], v, w) = \eta$ while χ has been produced on ξ can only happen, if the random walk S does not reach neither m^- nor m^+ in $[0, n]$. Hence for $\xi \in \bar{\Xi}$

$$\begin{aligned}&\mathbb{P}(T(\chi|[0,n], v, w) = \eta|\, \chi \text{ has been produced on } \xi) \\ &\quad = \mathbb{P}(S \text{ does not reach neither } m^- \text{ nor } m^+ \text{ in } [0,n]) \\ &\quad = \mathbb{P}(|S(i)| \leqslant \kappa n^{1/3} \text{for all } i = 0, \ldots, n)\end{aligned}$$

Now indeed there are positive constants $k_2, k_3 > 0$ such that

$$\mathbb{P}(|S(i)| \leqslant \kappa n^{1/3} \text{for all } i = 0, \ldots, n) \leqslant k_2 \mathrm{e}^{-k_2 n^{1/3}}.$$

To see why, just observe that by the local Central Limit Theorem, for each time interval $I[t_0, t_1]$ of length $n^{2/3}$ we have

$$\mathbb{P}(|S(i)| \leqslant \kappa n^{1/3} \text{for all } i \in I \| S(t_0)| \leqslant \kappa n^{1/3}) \leqslant k_4 < 1$$

for a positive constant $k_4 > 0$. Since there are $n^{1/3}$ disjoint intervals of length $n^{2/3}$ in $[0, n]$ this gives (by conditioning)

$$\begin{aligned}&\mathbb{P}(|S(i)| \leqslant \kappa n^{1/3} \text{for all } i = 0, \ldots, n) \\ &\quad = \mathbb{P}(|S(i)| \leqslant \kappa n^{1/3} \text{for all } i = 0, \ldots, n^{2/3}) \\ &\qquad \times \mathbb{P}(|S(i)| \leqslant \kappa n^{1/3} \text{for all } i = n^{2/3} + 1, \ldots 2n^{2/3} \| S(n^{2/3})| \leqslant \kappa n^{1/3}) \times \cdots \\ &\quad \leqslant k_4^{n^{1/3}} \leqslant k_2 \mathrm{e}^{-k_2 n^{1/3}}.\end{aligned}$$

This finishes the proof. $\square$

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